Unit 6 (continued)
Image Enhancement in the Frequency Domain

Background: Fourier Series
Any periodic signals can be viewed as weighted sum of sinusoidal signals with different frequencies

Frequency Domain: view frequency as an independent variable

Fourier Tr. and Frequency Domain

Fourier Tr.: \[ F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi xu} dx \]
Inv. Fourier Tr.: \[ f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi xu} du \]

1-D, Continuous case

Fourier Tr.: \[ F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x)e^{-j2\pi xu/M} \quad u = 0, \ldots, M-1 \]
Inv. Fourier Tr.: \[ f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi xu/M} \quad x = 0, \ldots, M-1 \]

F(u) can be written as
\[ F(u) = R(u) + jI(u) \quad \text{or} \quad F(u) = |F(u)|e^{-j\phi(u)} \]
where
\[ |F(u)| = \sqrt{R(u)^2 + I(u)^2} \quad \phi(u) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right) \]

Example of 1-D Fourier Transforms
Notice that the longer the time domain signal, The shorter its Fourier transform
Relation Between $\Delta x$ and $\Delta u$

For a signal $f(x)$ with $M$ points, let spatial resolution $\Delta x$ be space between samples in $f(x)$ and let frequency resolution $\Delta u$ be space between frequencies components in $F(u)$, we have

$$\Delta u = \frac{1}{100 \times 0.5} = 0.02 \text{ Hz}$$

This means that in $F(u)$ we can distinguish 2 frequencies that are apart by 0.02 Hertz or more.

2-Dimensional Discrete Fourier Transform (cont.)

$F(u,v)$ can be written as

$$F(u,v) = R(u,v) + jI(u,v) \quad \text{or} \quad F(u,v) = |F(u,v)|e^{-j\phi(u,v)}$$

where

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)} \quad \phi(u,v) = \tan^{-1}\left(\frac{I(u,v)}{R(u,v)}\right)$$

For the purpose of viewing, we usually display only the Magnitude part of $F(u,v)$

2-D DFT Properties (cont.)

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation of the inverse Discrete Fourier transform using a forward transform algorithm</td>
<td>Taking the complex conjugate and multiplying this result by $MN$ gives the desired inverse.</td>
</tr>
<tr>
<td>Convolution theorem'</td>
<td>$f(x,y) * R_s(x,y) = \frac{1}{MN} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(x-m,y-n)R_s(m,n)$</td>
</tr>
<tr>
<td>Correlation theorem'</td>
<td>$f(x,y) \otimes R_s(x,y) = \frac{1}{MN} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f^*(x-m,y-n)R_s(m,n)$</td>
</tr>
</tbody>
</table>
2-D DFT Properties (cont.)

Some useful FT pairs:

- Impulse \( \delta(x, y) \Rightarrow 1 \)
- Gaussian \( e^{-\pi^2 r^2} \Rightarrow e^{-\pi r^2} \)
- Rectangular \( \sin((\max(x,y)) \Rightarrow \sin((\max(x,y))) \)
- Cosine \( \cos(2\pi r x y) \Rightarrow 2\pi^2 r x y \)
- Sine \( \sin(2\pi r x y) \Rightarrow -i2\pi^2 r x y \)

Assumes that functions have been extended by zero padding.

Relation Between Spatial and Frequency Resolutions

\[
\Delta u = \frac{1}{M\Delta x} \quad \Delta v = \frac{1}{N\Delta y}
\]

where

- \( \Delta x = \text{spatial resolution in } x \text{ direction} \)
- \( \Delta y = \text{spatial resolution in } y \text{ direction} \)

(\( \Delta x \) and \( \Delta y \) are pixel width and height.)

\( \Delta u = \text{frequency resolution in } x \text{ direction} \)
\( \Delta v = \text{frequency resolution in } y \text{ direction} \)
\( N, M = \text{image width and height} \)

How to Perform 2-D DFT by Using 1-D DFT

Alternative method

From DFT:

\[
F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x)e^{-j2\pi xu/M}
\]

DFT repeats itself every \( N \) points (Period = \( N \)) but we usually display it for \( n = 0, \ldots, N-1 \)
Conventional Display for 1-D DFT

Time Domain Signal

Conventional Display for DFT: FFT Shift

FFT Shift: Shift center of the graph \( F(u) \) to 0 to get better Display which is easier to understand.

High frequency area
Low frequency area

Periodicity of 2-D DFT

2-D DFT: \( F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \)

For an image of size \( N \times M \) pixels, its 2-D DFT repeats itself every \( N \) points in \( x \)-direction and every \( M \) points in \( y \)-direction.

We display only in this range

Conventional Display for 2-D DFT

\( F(u,v) \) has low frequency areas at corners of the image while high frequency areas are at the center of the image which is inconvenient to interpret.

High frequency area
Low frequency area

2-D FFT Shift: Better Display of 2-D DFT

2-D FFT Shift is a MATLAB function: Shift the zero frequency of \( F(u,v) \) to the center of an image.

2-D FFT Shift (cont.): How it works
Notice that the longer the time domain signal, the shorter its Fourier transform.

Example of 2-D DFT

Notice that the direction of an object in spatial image and its Fourier transform are orthogonal to each other.

Example of 2-D DFT

Basic Concept of Filtering in the Frequency Domain

From Fourier Transform Property:

\[ g(x, y) = f(x, y) * h(x, y) \leftrightarrow F(u, v) * H(u, v) = G(u, v) \]

We can perform filtering process by using:

Multiplication in the frequency domain is easier than convolution in the spatial domain.

Filtering in the Frequency Domain with FFT shift

In this case, \( F(u, v) \) and \( H(u, v) \) must have the same size and have the zero frequency at the center.
Multiplication in Freq. Domain = Circular Convolution

\[ f(x) \rightarrow \text{DFT} \rightarrow F(u) \]
\[ h(x) \rightarrow \text{DFT} \rightarrow H(u) \]
\[ G(u) = F(u)H(u) \rightarrow \text{IDFT} \rightarrow g(x) \]

Multiplication of DFTs of 2 signals is equivalent to perform convolution in the spatial domain.

"Wrap around" effect

Original image

Multiplication in Freq. Domain = Circular Convolution

\[ H(u,v) \]
\[ \text{Gaussian Lowpass Filter with } D_0 = 5 \]

Filtered image (obtained using circular convolution)

Incorrect areas at image rims

Linear Convolution by using Circular Convolution and Zero Padding

\[ f(x) \rightarrow \text{Zero padding} \rightarrow \text{DFT} \rightarrow F(u) \]
\[ h(x) \rightarrow \text{Zero padding} \rightarrow \text{DFT} \rightarrow H(u) \]
\[ G(u) = F(u)H(u) \rightarrow \text{IDFT} \rightarrow g(x) \]

Concatenation

Zero padding area in the spatial Domain of the mask image (the ideal lowpass filter)

Padding zeros before DFT

Keep only this part

Filtered image

Linear Convolution by using Circular Convolution and Zero Padding

Zero padding area in the spatial Domain of the mask image (the ideal lowpass filter)

Filtered image

Only this area is kept.

Filtering in the Frequency Domain : Example

In this example, we set \( F(0,0) \) to zero which means that the zero frequency component is removed.

Note: Zero frequency = average intensity of an image
Filtering in the Frequency Domain: Example

Filter Masks and Their Fourier Transforms

Examples of Ideal Lowpass Filters

Results of Ideal Lowpass Filters

The smaller $D_0$, the more high frequency components are removed.

Ringing effect can be obviously seen!
How ringing effect happens

\[ H(u, v) = \begin{cases} 1 & D(u, v) \leq D_0 \\ 0 & D(u, v) > D_0 \end{cases} \]

Ideal Lowpass Filter with \( D_0 = 5 \)

Surface Plot

Abrupt change in the amplitude

How ringing effect happens (cont.)

Spatial Response of Ideal Lowpass Filter with \( D_0 = 5 \)

Ripples that cause ringing effect

How ringing effect happens (cont.)

Spatial Masks of the Butterworth Lowpass Filters

There is less ringing effect compared to those of ideal lowpass filters!

Butterworth Lowpass Filter

Transfer function

\[ H(u, v) = \frac{1}{1 + \left(\frac{D(u, v)}{D_0}\right)^{2N}} \]

Where \( D_0 = \) Cut off frequency, \( N = \) filter order.

Results of Butterworth Lowpass Filters

Spatial Masks of the Butterworth Lowpass Filters

Some ripples can be seen.
**Gaussian Lowpass Filter**

**Transfer function**

\[ H(u, v) = e^{-D v^2 + 2D u^2} \]

Where \( D_0 \) is spread factor.

Note: the Gaussian filter is the only filter that has no ripple and hence no ringing effect.

**Results of Gaussian Lowpass Filters**

No ringing effect!

**Application of Gaussian Lowpass Filters**

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "90" as 1990 rather than the year 2000.

The GLPF can be used to remove jagged edges and "repair" broken characters.

**Application of Gaussian Lowpass Filters (cont.)**

Remove wrinkles

**Application of Gaussian Lowpass Filters (cont.)**

Remove artifact lines: this is a simple but crude way to do it!
**Highpass Filters**

\[ H_{hp} = 1 - H_{lp} \]

**Butterworth Highpass Filters**

Transfer function

\[ H(u, v) = \frac{1}{1 + [D_0 / D(u,v)]^{2N}} \]

Where \( D_0 \) = Cut off frequency, \( N \) = filter order.

**Gaussian Highpass Filters**

Transfer function

\[ H(u, v) = 1 - e^{-D(u,v)^2/D_0^2} \]

Where \( D_0 \) = spread factor.

**Spatial Responses of Highpass Filters**

Gaussian highpass filter with \( D_0 = 5 \)

Spatial responses of the Gaussian highpass filter with \( D_0 = 5 \)

Ripples
Results of Ideal Highpass Filters

RINGING EFFECT CAN BE OBVIOUSLY SEEN!

Results of Butterworth Highpass Filters

Results of Gaussian Highpass Filters

Laplacian Filter in the Frequency Domain

From Fourier Transform Property:
\[ \frac{d^2 f(x)}{dx^2} \Rightarrow (ju)^2 F(u) \]

Then for Laplacian operator
\[ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \Rightarrow -(u^2 + v^2)F(u, v) \]

We get
\[ \nabla^2 \Rightarrow -(u^2 + v^2) \]

Spatial Domain
\[ f_{lp}(x, y) = f(x, y) - f_{hp}(x, y) \]
\[ f_{hp}(x, y) = Af(x, y) - f_{lp}(x, y) \]
\[ f_{hp}(x, y) = (A-1)f(x, y) + f(x, y) - f_{lp}(x, y) \]
\[ f_{hp}(x, y) = (A-1)f(x, y) + f_{hp}(x, y) \]

Frequency Domain Filter
\[ H_{hp}(u, v) = 1 - H_{lp}(u, v) \]
\[ H_{hp}(u, v) = (A-1) + H_{hp}(u, v) \]
Sharpening Filtering in the Frequency Domain (cont.)

1. \( P \)
2. \( \nabla^2 P \)
3. \( P - \nabla^2 P \)

High Frequency Emphasis Filtering

\[ H_{\text{hp}}(u, v) = a + bH_{\text{sp}}(u, v) \]

Homomorphic Filtering:

- If the image model is based on illumination-reflectance, then frequency domain procedures are not as easy to perform.
- The main reason is that illumination and reflectance components of the model are not separable.
- To be able to improve appearance of an image by simultaneous brightness range compression and contrast enhancement it is necessary to separate the two components.

Homomorphic Filtering:

- The illumination component of an image is generally characterized by slow spatial variation.
- The reflectance component of an image tends to vary abruptly.
- These characteristics lead to associating the low frequencies of the Fourier transform of the natural log of an image with illumination and high frequencies with reflectance.
- A good deal of control can be gained over the illumination and reflectance components with a homomorphic filter.
- For homomorphic filter to be effective, it needs to affect the low- and high-frequency components of the Fourier transform in different ways.
- To compress the dynamic range of an image, the low frequency components of the Fourier transform ought to be attenuated to some degree.
- On the other hand, to enhance the contrast, the high frequency components of the Fourier transform ought to be magnified.
**Homomorphic Filtering**

\[ f(x, y) \xrightarrow{\text{DFT}} F(u, v) \xrightarrow{H(u, v)} \xrightarrow{\text{DFT}^{-1}} g(x, y) \xrightarrow{\exp} \tilde{f}(x, y) \]

**FIGURE 4.31**
Homomorphic filtering approach for image enhancement.

**FIGURE 4.32**
Cross section of a circularly symmetric filter function. \( D(u, v) \) is the distance from the origin of the centered transform.

**Correlation Application: Object Detection**

More details in the room can be seen!