Unit 7

Detection of known signals in noise

Assume that in each time slot of duration T seconds, one of the M possible signals $S_1(t)$, $S_2(t)$. . . . . . . $S_M(t)$ is transmitted with equal probability of 1/M. Then for an AWGN channel a possible realization of sample function x(t), of the received random process X(t) is given by

$$x(t) = S_i(t) + w(t) \quad 0 \le t \le T$$
$$i = 1,2,3,.............,M$$

where w(t) is sample function of the white Gaussian noise process W(t), with zero mean and PSD $N_0/2$. The receiver has to observe the signal x(t) and make a **best estimate** of the transmitted signal $s_i(t)$ or equivalently symbol m_i

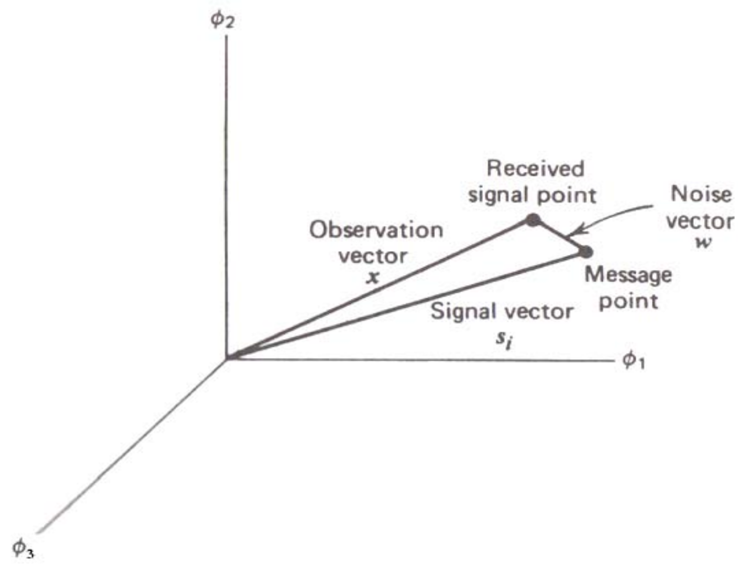
The transmitted signal si(t), i= 1to M , is applied to a bank of correlators, with a common input and supplied with an appropriate set of N orthonormal basic functions, the resulting correlator outputs define the signal vector $\mathbf{S_i}$. knowing **Si** is as good as knowing the transmitted signal Si(t) itself, and vice versa. We may represents $s_i(t)$ by a point in a Euclidean space of dimensions $N \le M$. . Such a point is referred as transmitted signal point or message point. The collection of M message points in the N Euclidean space is called a **signal constellation.**

When the received signal x(t) is applied to the bank o N correlators , the output of the correlator define a new vector **x** called observation vector. this vector **x** differs from the signal vector s_i by a random noise vector **w**

$$x = S_i + w \qquad i = 1,2,3,.........,M$$

The vectors **x** and **w** are sampled values of the random vectors **X** and **W** respectively. the noise vector **w** represents that portion of the noise w(t) which will interfere with the detected process.

Based on the observation vector **x**, we represent the received signal s(t)by a point in the same Euclidean space, we refer this point as **received signal point**. The relation between them is as shown in the fig



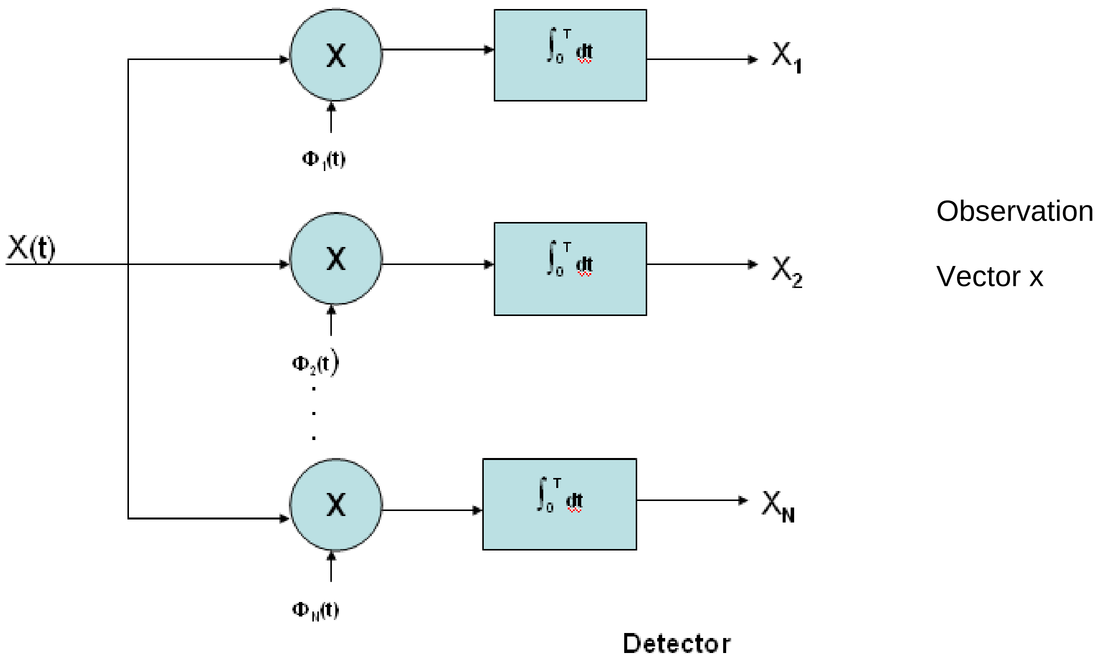
<u>Fig:</u> Illustrating the effect of noise perturbation on location of the received signal point

In the detection problem , the observation vector x is given, we have to perform a mapping from **x** to an estimate of the transmitted symbol, in away that would minimize the average probability of symbol error in the decision. The **maximum likelihood detector** provides solution to this problem.

Optimum transmitter & receiver

- Probability of error depends on signal to noise ratio
- As the SNR increases the probability of error decreases
- An optimum transmitter and receiver is one which maximize the SNR and minimize the probability of error.

Correlative receiver

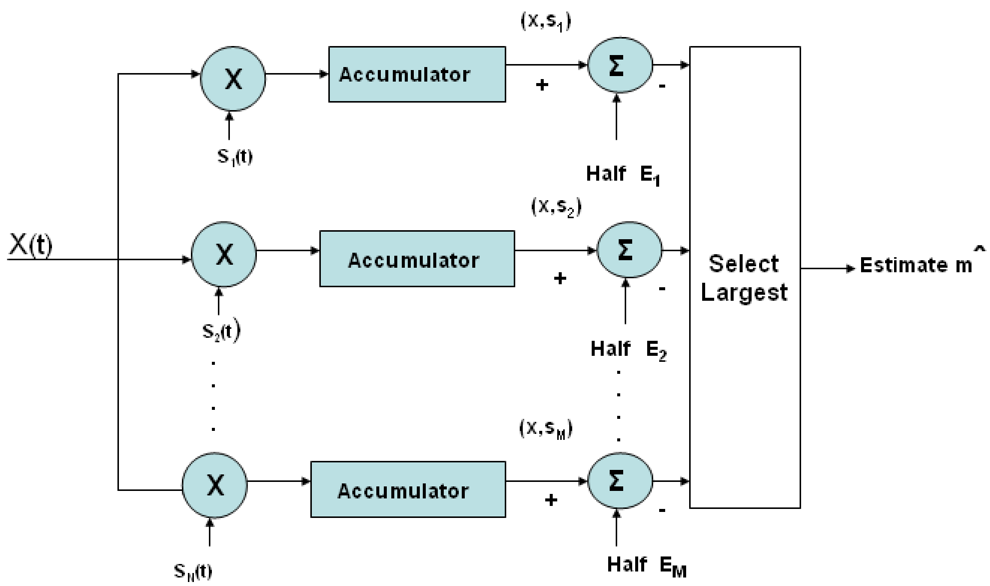


For an AWGN channel and for the case when the transmitted signals are equally likely, the optimum receiver consists of two subsystems

1) .Receiver consists of a bank of M product-integrator or correlators

$\Phi_1(t)$,$\Phi_2(t)$ …….$\Phi_M(t)$ orthonormal function

The bank of correlator operate on the received signal x(t) to produce observation vector x



Vector Receiver

2). Implemented in the form of maximum likelihood detector that

operates on observation vector **x** to produce an estimate of the transmitted symbol

m_i i = 1 to M, in a way that would minimize the average probability of symbol error.

The N elements of the observation vector **x** are first multiplied by the corresponding N elements of each of the M signal vectors s_1, s_2… s_M , and the resulting products are successively summed in accumulator to form the corresponding set of

Inner products {(x, s_k)} k= 1, 2 ..M. The inner products are corrected for the fact that the transmitted signal energies may be unequal. Finally,

the largest in the resulting set of numbers is selected and a corresponding decision on the transmitted message made.

The optimum receiver is commonly referred as a **correlation receiver**

MATCHED FILTER

Science each of t he orthonormal basic functions are $\Phi_1(t)$,$\Phi_2(t)$ …….$\Phi_M(t)$ is assumed to be zero outside the interval $0 \le t \le T$. we can design a linear filter with impulse response $h_j(t)$, with the received signal x(t) the fitter output is given by the convolution integral

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau) h_j(t-\tau) d\tau$$

Suppose the impulse response of the system is

$$\mathbf{h}_j(t) = \phi_j(T - t)$$

Then the filter output is

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)\phi_j(T - t + \tau)d\tau$$

sampling this output at time t = T, we get

$$y_j(T) = \int_{-\infty}^{\infty} x(\tau)\phi_j(\tau)d\tau$$

$\Phi_j(t)$ is zero outside the interval $0 \le t \le T$,we get

$$y_j(T) = \int_{0}^{T} x(\tau)\phi_j(\tau)d\tau$$

$y_j(t) = x_j$

where x_j is the j^{th} correlator output produced by the received signal x(t).

A filter whose impulse response is time-reversed and delayed version of the input signal

$\phi_j(t)$ is said to be matched to $\phi_j(t)$. correspondingly , the optimum receiver based on this is referred as the **matched filter receiver**.

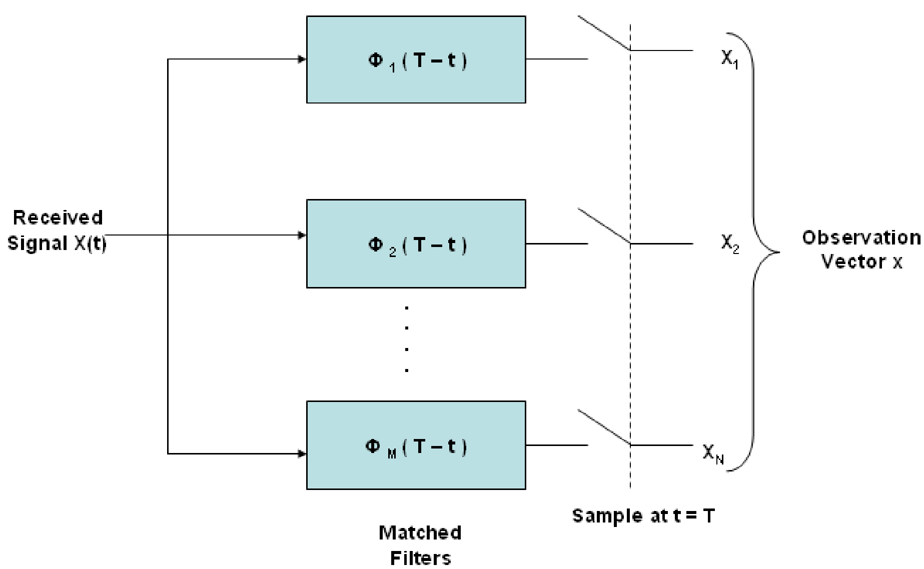
For a matched filter operating in real time to be physically realizable, it must be causal.

For causal system

$$h_j(t) = 0 \qquad t < 0$$

causality condition is satisfied provided that the signal $\phi_j(t)$ is zero outside the interval

$$0 \le t \le T$$



Detector Part Of Matched Filter Receiver

<u>Maximization of output SNR in matched filter</u>

Let

x(t) = input signal to the matched filter

h(t) = impulse response of the matched filter

w(t) =white noise with power spectral density $N_o/2$

$\phi(t)$ = known signal

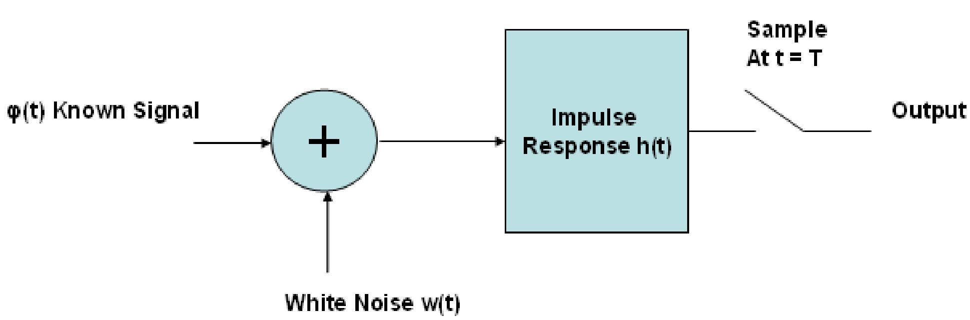
Input to the matched filter is given by

$$x(t) = \phi(t) + w(t) \qquad 0 \le t \le T$$

science the filter is linear , the resulting output y(t) is given by

$$y(t) = \phi_0(t) + n(t)$$

where $\phi_0(t)$ and n(t) are produced by the signal and noise components of the input x(t).



The signal to noise ratio at the output of the matched filter at t = T is

$$(SNR)_0 = \frac{|\phi_0(T)|^2}{E[n^2(t)]} \cdots\cdots\cdots\cdots(6.13)$$

aim is to find the condition which maximize the SNR

let

$$\phi_0(t) \leftrightarrow \phi_0(f)$$
$$h(t) \leftrightarrow H(f)$$

are the Fourier transform pairs, hence the output signal $\phi_0(t)$ is given by

$$\phi_0(t) = \int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi ft)df$$

output at t = T is

$$|\phi_0(T)|^2 = \left|\int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi fT)df\right|^2 \cdots\cdots\cdots(6.14)$$

For the receiver input noise with psd $N_o/2$ the receiver output noise psd is given by

$$S_N(f) = \frac{N_0}{2}|H(f)|^2 \quad \cdots\cdots\cdots(6.15)$$

and the noise power is given by

$$E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)df$$
$$= \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df \cdots\cdots\cdots(6.16)$$

substituting the values of eqns 6.14 & 6.15 in 6.13 we get

$$(SNR)_0 = \frac{\left|\int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi fT)df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df} \cdots\cdots\cdots(6.17)$$

using Schwarz's inequality

$$\left|\int_{-\infty}^{\infty} X_1(f)X_2(f)df\right|^2 \le \int_{-\infty}^{\infty}|X_1(f)|^2 df \int_{-\infty}^{\infty}|X_2(f)|^2 df \cdots\cdots\cdots(6.18)$$

Eqn 6.16 is equal when $X_1(f) = kX_2^*(f)$

let $X_1(f) = H(f)$

& $X_2(f) = \Phi(f)\exp(j2\pi fT)$

under equality condition

$H(f) = K\Phi^*(f)\exp(-j2\pi fT)\cdots\cdots\cdots(6.19)$

Thus substituting in 6.16 we get the value

$$\left| \int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi fT)df \right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |\Phi(f)|^2 df$$

substituting in eqn 6,17 and simplifying

$$(SNR)_0 \le \frac{2}{N_0} \int_{-\infty}^{\infty} |\Phi(f)|^2 df$$

Using Rayleigh's energy theorem

$$\int_{-\infty}^{\infty} |\phi(t)|^2 dt = \int_{-\infty}^{\infty} |\Phi(f)|^2 df = E, \quad energy\ of\ the\ signal$$

$$(SNR)_{0,\max} = \frac{2E}{N_0} \cdots\cdots\cdots (6.20)$$

Under maximum SNR condition, the transfer function is given by (k=1), eqn 6.19

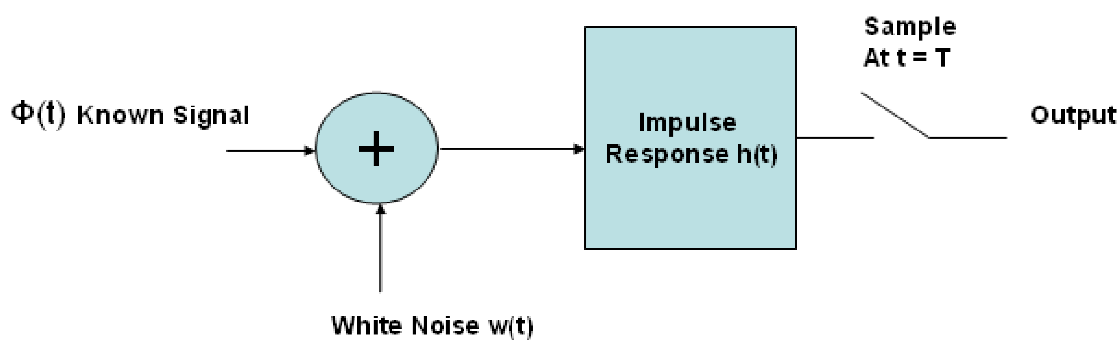
$$H_{opt}(f) = \Phi^*(f)\exp(-j2\pi fT)$$

The impulse response in time domain is given by

$$h_{opt}(t) = \int_{-\infty}^{\infty} \Phi(-f)\exp[-j2\pi f(T-t)]\exp(j2\pi ft)df$$
$$= \phi(T-t)$$

Thus the impulse response is folded and shifted version of the input signal $\phi(t)$

MATCHED FILTER



Φ(t) = input signal

h(t) = impulse response

W(t) =white noise

The impulse response of the matched filter is time-reversed and delayed version of the input signal

$$\mathbf{h}\ (t) = \phi\ (T - t)$$

For causal system

$$h_j(t) = 0 \qquad t < 0$$

<u>Matched filter properties</u>

PROPERTY 1

The spectrum of the output signal of a matched filter with the matched signal as input is, except for a time delay factor, proportional to the energy spectral density of the input signal.

let $\Phi_0(f)$ denotes the Fourier transform of the filter output $\Phi_0(t)$, hence

$$\begin{aligned}\Phi_0(f) &= H_{opt}(f)\Phi(f) \quad sustituting \quad from\ 6.19 \\ &= \Phi^*(f)\Phi(f)\exp(-j2\pi fT) \\ &= |\Phi(f)|^2 \exp(-2j\pi fT) \cdots\cdots\cdots (6.21)\end{aligned}$$

PROPERTY 2

The output signal of a Matched Filter is proportional to a shifted version of the autocorrelation function of the input signal to which the filter is matched.

The autocorrelation function and energy spectral density of a signal forms the Fourier transform pair, thus taking inverse Fourier transform for eqn 6.21

$$\phi_0(t) = R_\phi(t - T)$$

At time t = T

$$\phi_0(T) = R_\phi(0) = E$$

where E is energy of the signal

PROPERTY 3

The output Signal to Noise Ratio of a Matched filter depends only on the ratio of the signal energy to the power spectral density of the white noise at the filter input.

SNR at the output of matched filter is eqn 6.13

$$(SNR)_0 = \frac{|\phi_0(T)|^2}{E[n^2(t)]} \cdots\cdots\cdots (6.22)$$

output of matched filter is

$$\phi_0(t) = \int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi ft)df$$

signal power at t = T

$$\left|\phi_0(T)\right|^2 = \left|\int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi fT)df\right|^2$$

noise psd at the output of receiver is

$$S_N(f) = \frac{N_0}{2}\left|H(f)\right|^2$$

noise power is

$$E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df$$

substituting the values in 6.22 we get

$$(SNR)_0 = \frac{\left|\int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi fT)df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df} \cdots\cdots\cdots (6.23)$$

using Schwarz's inequality

$$\left|\int_{-\infty}^{\infty} X_1(f)X_2(f)df\right|^2 \le \int_{-\infty}^{\infty}|X_1(f)|^2 df \int_{-\infty}^{\infty}|X_2(f)|^2 df \cdots\cdots\cdots(6.24)$$

Eqn 6.24 is equal when $X_1(f) = kX_2^*(f)$

let $X_1(f) = H(f)$

& $X_2(f) = \Phi(f)\exp(j2\pi fT)$

under equality condition

$$H(f) = K\Phi^*(f)\exp(-j2\pi fT)\cdots\cdots\cdots(6.25)$$

Thus substituting in 6.24 we get the value

$$\left|\int_{-\infty}^{\infty} H(f)\Phi(f)\exp(j2\pi fT)df\right|^2 \le \int_{-\infty}^{\infty}|H(f)|^2 df \int_{-\infty}^{\infty}|\Phi(f)|^2 df$$

substituting in eqn 6,23 and simplifying

$$(SNR)_0 \le \frac{2}{N_0}\int_{-\infty}^{\infty}|\Phi(f)|^2 df$$

Using Rayleigh's energy theorem

$$\int_{-\infty}^{\infty}|\phi(t)|^2 dt = \int_{-\infty}^{\infty}|\Phi(f)|^2 df = E, \quad \textit{energy of the signal}$$

$$(SNR)_{0,\max} = \frac{2E}{N_0}\cdots\cdots\cdots(6.26)$$

PROPERTY 4

The Matched Filtering operation may be separated into two matching conditions; namely spectral phase matching that produces the desired output peak at time T, and the spectral amplitude matching that gives this peak value its optimum signal to noise density ratio.

In polar form the spectrum of the signal $\phi(t)$ being matched may be expressed as

$$\Phi(f) = |\Phi(f)| \exp[j\theta(f)]$$

where $|\Phi(f)|$ is magnitude spectrum and $\theta(f)$ is phase spectrum of the signal.

The filter is said to be spectral phase matched to the signal $\phi(t)$ if the transfer function of the filter is defined by

$$H(f) = |H(f)| \exp[-j\theta(f) - j2\pi fT]$$

The output of such a filter is

$$\phi_0^{'}(t) = \int_{-\infty}^{\infty} H(f)\,\Phi(f)\exp(j2\pi ft)df$$

$$= \int_{-\infty}^{\infty} |H(f)||\Phi(f)|\exp[j2\pi f(t-T)]df$$

The product $|H(f)||\Phi(f)|$ is real and non negative.

The spectral phase matching ensures that all the spectral components of the output add constructively at t = T , there by causing the output to attain its maximum value.

$$\phi_0^{'}(t) \le \phi_0^{'}(T) = \int_{-\infty}^{\infty} |\Phi(f)||H(f)|df$$

For spectral amplitude matching

$$|H(f)| = |\Phi(f)|$$

Problem-1:

Consider the four signals $s_1(t)$, $s_2(t)$, $s_3(t)$ and $s_4(t)$ as shown in the fig-P1.1 .

Use Gram-Schmidt Orthogonalization Procedure to find the orthonormal basis for this set of signals. Also express the signals in terms of the basis functions.

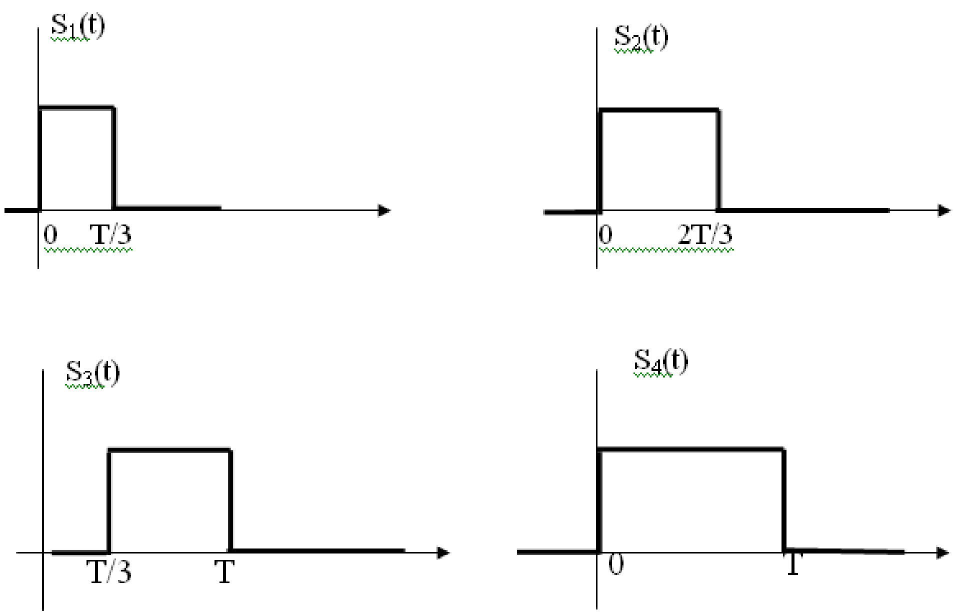


Fig-P1.1: Signals for the problem -1.

Solution:

Given set is not linearly independent because $s_4(t) = s_1(t) + s_3(t)$

Step-1: Energy of the signal $s_1(t)$

$$E_1 = \int_0^T s_1^2(t)\, dt = \frac{T}{3}$$

First basis function

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \sqrt{3/T} \quad for\ 0 \le t \le T/3$$

Step-2: Coefficient s_{21}

$$s_{21} = \int_0^T s_2(t)\phi_1(t)\, dt = \sqrt{T/3}$$

Energy of $s_2(t)$

$$E_2 = \int_0^T s_2^2(t)\, dt = \frac{2T}{3}$$

Second Basis function

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} = \sqrt{3/T} \quad for\ T/3 \le t \le 2T/3$$

Step-3: Coefficient $s_{31:}$

$$s_{31} = \int_0^T s_3(t)\phi_1(t)\, dt = 0$$

Coefficient s_{32}

$$s_{32} = \int_0^T s_3(t)\phi_2(t)\, dt = \sqrt{T/3}$$

Intermediate function

$$g_3(t) = s_3(t) - s_{31}\Phi_1(t) - s_{32}\Phi_2(t)$$

$g_3(t) = 1$ for $2T/3 \le t \le T/3$

Third Basis function

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)\,dt}} = \sqrt{3/T} \quad for \;\; 2T/3 \le t \le T$$

The corresponding orthonormal functions are shown in the figure-P1.2.

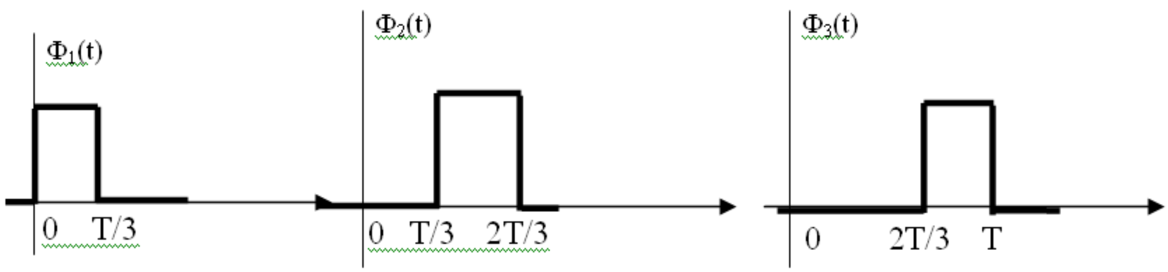


Fig-P1.2: Orthonormal functions for the Problem-1

Representation of the signals

$$S_1(t) = \sqrt{T/3}\;\phi_1(t)$$

$$S_2(t) = \sqrt{T/3}\;\phi_1(t) + \sqrt{T/3}\;\phi_2(t)$$

$$S_3(t) = \sqrt{T/3}\;\phi_2(t) + \sqrt{T/3}\;\phi_3(t)$$

$$S_4(t) = \sqrt{T/3}\;\phi_1(t) + \sqrt{T/3}\;\phi_2(t) + \sqrt{T/3}\;\phi_3(t)$$

PROBLEM-2:

Consider the THREE signals $s_1(t)$, $s_2(t)$ and $s_3(t)$ as shown in the fig P2.1. Use Gram-Schmidt Orthogonalization Procedure to find the orthonormal basis for this set of signals. Also express the signals in terms of the basis functions.

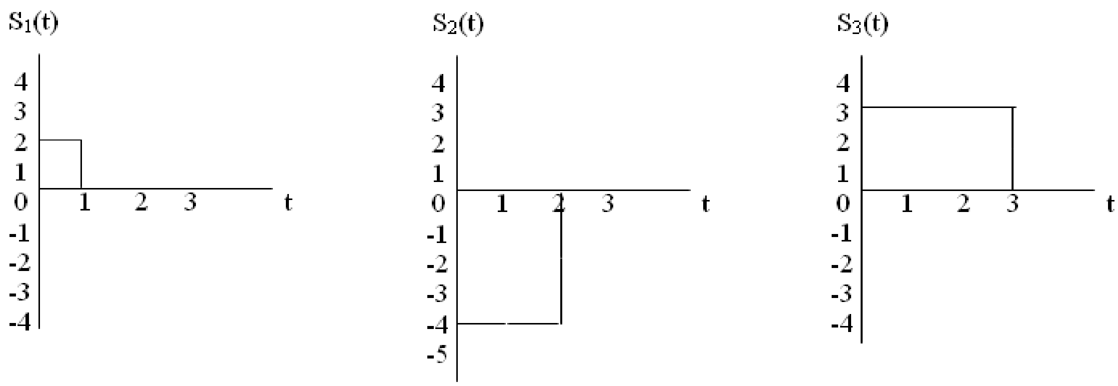


Fig-P2.1: Signals for the problem -2.

Solution: The basis functions are shown in fig-P2.2.

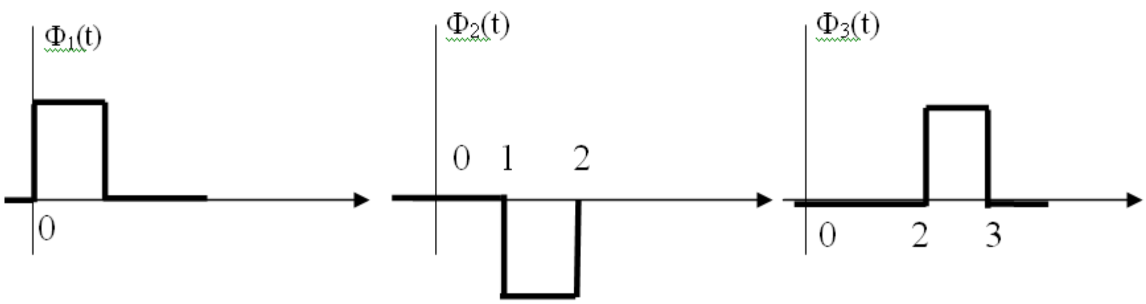


Fig-P2.2: Orthonormal functions for the Problem-2

Correspondingly the representation of the signals are:

$$S_1(t) = 2\ \phi_1(t)$$

$$S_2(t) = -4\ \phi_1(t)\ +4\ \phi_2(t)$$

$$S_3(t) = 3\ \phi_1(t)\ -3\ \phi_2(t) + 3\ \phi_3(t)$$

PROBLEM-3:

Consider the signal s(t) in fig-P3.1

a) Determine the impulse response of a filter matched to this signal and sketch it as a function of time.

b) Plot the matched filter output as a function of time.

c) What is Peak value of the output?

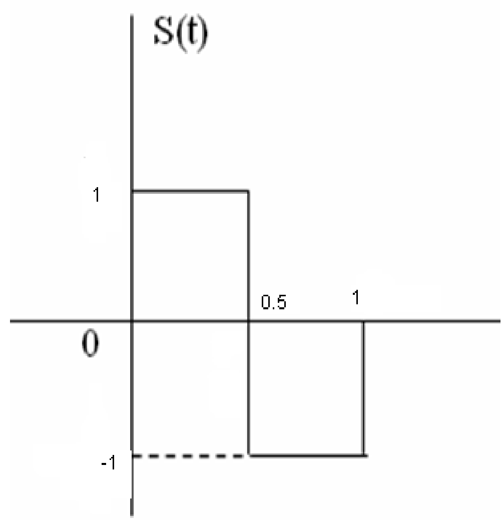


Fig P3.1

Solution:

The impulse response of the matched filter is time-reversed and delayed version of the input signal, $h(t) = s(T-t)$ and the output of the filter, $y(t) = x(t) * h(t)$.

Given $s(t) = +1$ for $0 < t < 0.5$
-1 for $0.5 < t < 1$.

(a) With $T = 1$, the impulse

response $h(t)$ is

$h(t) = -1$ for $0 < t < 0.5$

$+1$ for $0.5 < t < 1$.

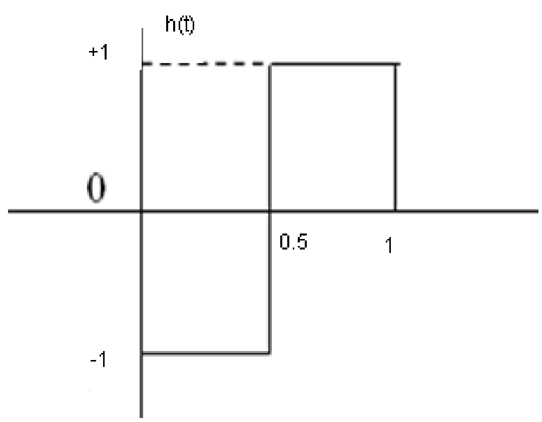


Fig. P3.2

(b) The output of the filter y(t) is obtained by convolving the input s(t) and the impulse response h(t). The corresponding output is shown in the fig. P3.3.

(c) The peak value of the output is 1.0 unit.

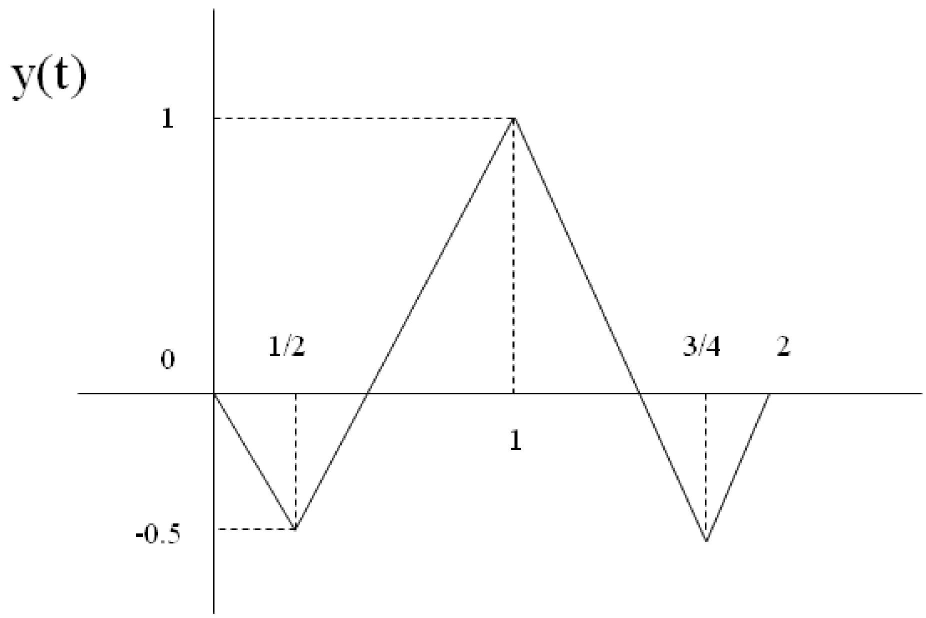


Fig. P3.3

Assignment Problem:

Specify a matched filter for the signal S1(t) shown in Fig.-P4.1 Sketch the output of the filter matched to the signal S_1(t) is applied to the filter input.

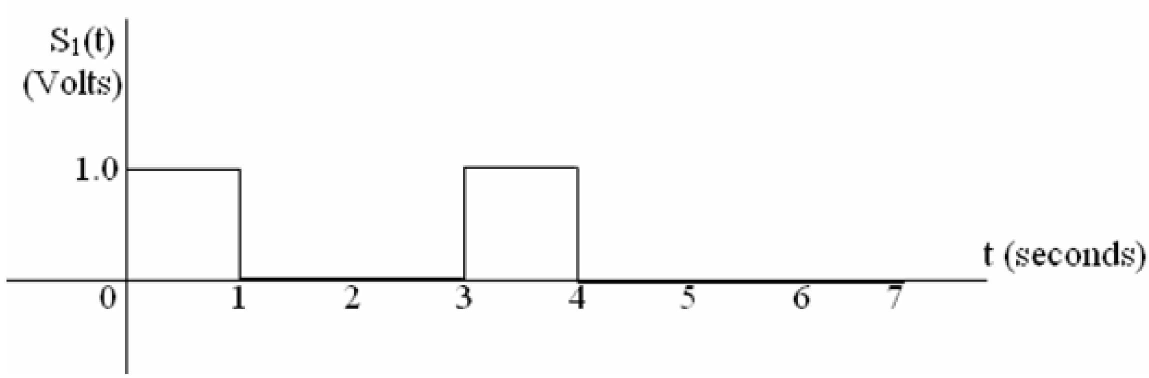


Fig P4.1