Chapter 6

DETECTION AND ESTIMATION:

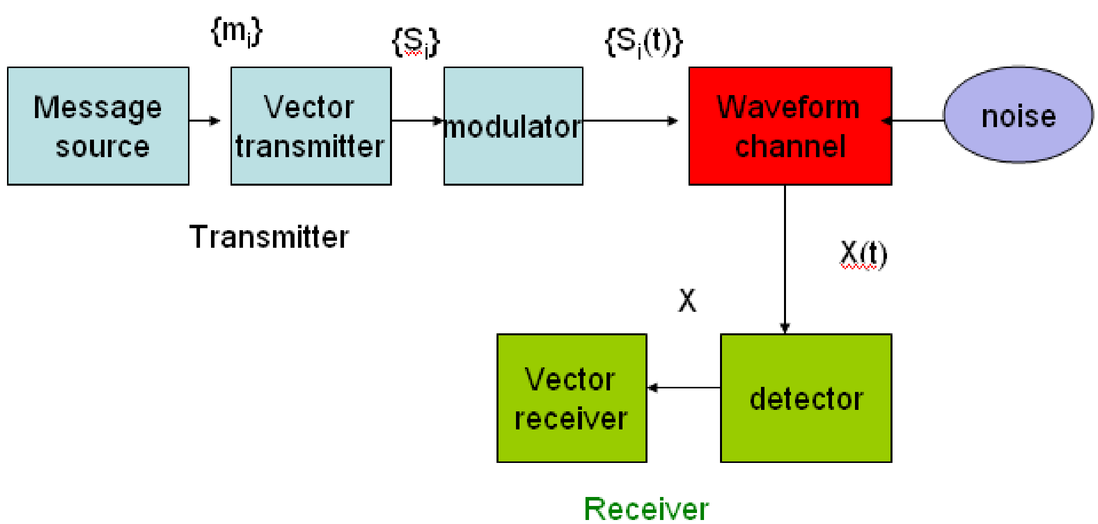
Fundamental issues in digital communications are

1. Detection and
2. Estimation

Detection theory: It deals with the design and evaluation of decision – making processor that observes the received signal and guesses which particular symbol was transmitted according to some set of rules.

Estimation Theory: It deals with the design and evaluation of a processor that uses information in the received signal to extract estimates of physical parameters or waveforms of interest.
The results of detection and estimation are always subject to errors

Model of digital communication system



Consider a source that emits one symbol every T seconds, with the symbols belonging to an alphabet of M symbols which we denote as $m_1, m_2, \ldots \ldots m_M$.
We assume that all M symbols of the alphabet are equally likely. Then

$$p_i = p(m_i \; emitted)$$
$$= \frac{1}{M} \quad for\, all \;\; i$$

The output of the message source is presented to a vector transmitter producing vector of real number

$$S_i = \begin{bmatrix} S_{i1} \\ S_{i2} \\ . \\ . \\ . \\ S_{iN} \end{bmatrix} i = 1, 2, ..., M$$

Where the dimension $N \leq M$.

The modulator then constructs a distinct signal $s_i(t)$ of duration T seconds. The signal $s_i(t)$ is necessarily of finite energy.

The Channel is assumed to have two characteristics:

- Channel is linear, with a bandwidth that is large enough to accommodate the transmission of the modulator output $s_i(t)$ without distortion.
- The transmitted signal $s_i(t)$ is perturbed by an additive, zero-mean, stationary, white, Gaussian noise process.

such a channel is referred as AWGN (additive white Gaussian noise) channel

GRAM – SCHMIDT ORTHOGONALIZATION PROCEDURE:

In case of Gram-Schmidt Orthogonalization procedure, any set of ‘M’ energy signals {Si(t)} can be represented by a linear combination of ‘N’ orthonormal basis functions where $N \leq M$. That is we may represent the given set of real valued energy signals $S_1(t)$, $S_2(t)$. . . . . . . $S_M(t)$ each of duration T seconds in the form

$$S_1(t) = S_{11}\phi_1(t) + S_{12}\phi_2(t). \dots\dots + S_{1N}\phi_N(t)$$

$$S_2(t) = S_{21}\phi_1(t) + S_{22}\phi_2(t). \dots\dots + S_{2N}\phi_N(t)$$

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$$S_M(t) = S_{M1}\phi_1(t) + S_{M2}\phi_2(t). \dots\dots + S_{MN}\phi_N(t)$$

$$S_i(t) = \sum_{j=1}^{N} S_{ij}\phi_j(t) \begin{cases} 0 \leq t \leq T \\ i = 1,2,3......M \end{cases} \cdots\cdots(6.1)$$

Where the Co-efficient of expansion are defined by

$$S_{ij}(t) = \int_0^T S_i(t)\phi_j(t)dt \begin{cases} i = 1,2,3. \ldots M \\ j = 1,2,3\ldots\ldots N \cdots\cdots (6.2) \end{cases}$$

The basic functions $\phi_1(t), \phi_2(t) \ldots\ldots\ldots \phi_N(t)$ are orthonormal by which

$$\int_0^T \phi_i(t)\phi_j(t)\,dt = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \cdots\cdots (6.3) \end{cases}$$

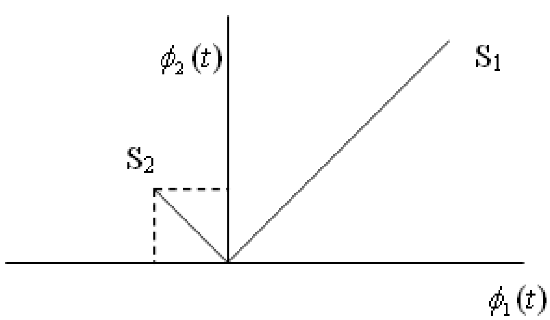
The co-efficient S_{ij} may be viewed as the j^{th} element of the N – dimensional Vector S_i

Therefore $$S_i = \begin{bmatrix} S_{i1} \\ S_{i2} \\ , \\ , \\ , \\ , \\ S_{iN} \end{bmatrix}$$ i = 1,2,3 . . . . . . M

Let $S_1 = 3\phi_1(t) + 4\phi_2(t) \qquad S_2 = -\phi_1(t) + 2\phi_2(t)$

Vector

$$S_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad S_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



Geometric interpretation of signal:

Using N orthonormal basis functions we can represent M signals as

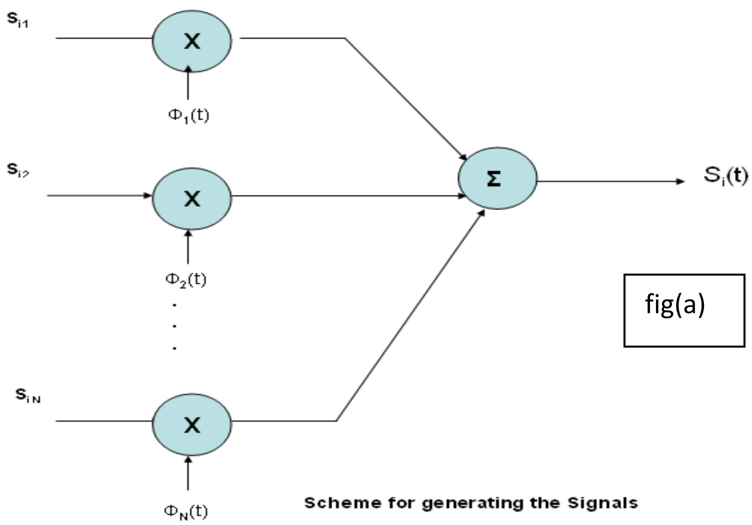
$$S_i(t) = \sum_{j=1}^{N} S_{ij}\phi_j(t) \quad 0 \leq t \leq T \qquad i = 1, 2, \ldots\ldots, M \cdots\cdots (6.4)$$

Coefficients are given by

$$S_{ij} = \int_0^T S_i(t)\phi_j(t)\,dt \quad i = 1, 2, \ldots.., M$$

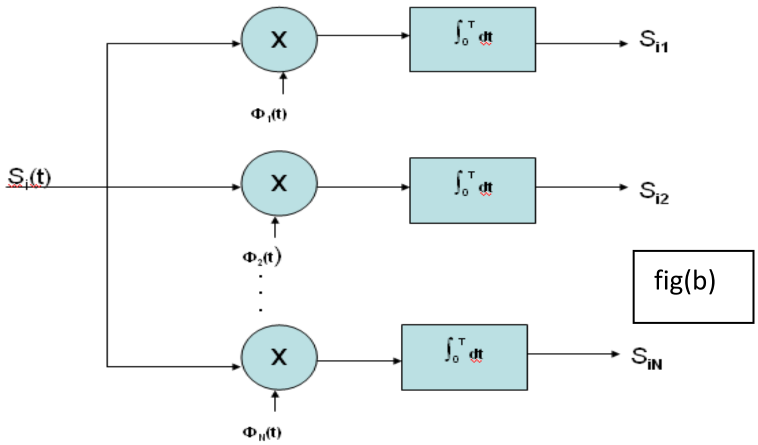
$$j = 1, 2, \ldots.., N \quad \cdots\cdots\cdots(6.5)$$

Given the set of coefficients $\{s_{ij}\}$, j= 1, 2, ….N operating as input we may use the scheme as shown in fig(a) to generate the signal $s_i(t)$ i = 1 to M. It consists of a bank of N multipliers, with each multiplier supplied with its own basic function, followed by a summer.



fig(a)

conversely given a set of signals $s_i(t)$ i = 1 to M operating as input we may use the scheme shown in fig (b) to calculate the set of coefficients $\{s_{ij}\}$, j= 1, 2, ….N



Scheme for generating the set of coefficients

$$S_i = \begin{bmatrix} S_{i1} \\ S_{i2} \\ . \\ . \\ . \\ S_{iN} \end{bmatrix} \quad i = 1, 2, \ldots.., M$$

The vector s_i is called **signal vector**

We may visualize signal vectors as a set of M points in an N dimensional Euclidean space, which is also called **signal space**

The squared-length of any vector s_i is given by inner product or the dot product of s_i

$$\|S_i\|^2 = (S_i, S_i) = \sum_{j=1}^{N} S_{ij}^2$$

Where s_{ij} are the elements of s_i

Two vectors are **orthogonal if their inner product is zero**

The energy of the signal is given by

$$E_i = \int_0^T S_i^2(t)\,dt$$

substituting the value $s_i(t)$ from equation 6.1

$$E_i = \int_0^T [\sum_{j=1}^{N} S_{ij}\phi_j(t)]\ [\sum_{k=1}^{N} S_{ik}\phi_k(t)]\,dt$$

interchanging the order of summation and integration

$$E_i = \sum_{j=1}^{N}\sum_{k=1}^{N} S_{ij}S_{ik}\int_0^T \phi_j(t)\phi_k(t)\,dt$$

since $\phi_j(t)$ forms an orthonormal set, the above equation reduce to

$$E_i = \sum_{j=1}^{N} S_{ij}^2$$

this shows that the energy of the signal $s_i(t)$ is equal to the squared-length of the signal vector s_i
The **Euclidean distance** between the points represented by the signal vectors s_i and s_k is

$$\|S_i - S_k\|^2 = \sum_{j=1}^{N} (S_{ij} - S_{kj})^2$$

$$= \int_0^T [S_i(t) - S_k(t)]^2 \, dt$$

Response of bank of correlators to noisy input

Received Signal X(t) is given by

$$X(t) = S_i(t) + W(t) \qquad 0 \le t \le T$$
$$i = 1,2,3\ldots\ldots, M \cdots\cdots\cdots (6.6)$$

where W(t) is AWGN with Zero Mean and PSD $N_0/2$
Output of each correlator is a random variable defined by

$$X_j = \int_o^T X(t)\phi_j(t)dt$$
$$= S_{ij} + W_j \qquad j = 1,2,\ldots\ldots..N \cdots\cdots\cdots\cdots (6.7)$$

The first Component S_{ij} is deterministic quantity contributed by the transmitted signal $S_i(t)$, it is defined by

$$S_{ij} = \int_0^T S_i(t)\phi_j(t)dt \cdots\cdots\cdots\cdots (6.8)$$

The second Component W_j is a random variable due to the presence of the noise at the input, it is defined by

$$W_j = \int_o^T W(t)\phi_j(t)dt \cdots\cdots\cdots\cdots (6.9)$$

let $X'(t)$ is a new random variable defined as

$$X'(t) = X(t) - \sum_{j=1}^{N} X_j \phi_j(t) \cdots\cdots\cdots\cdots (6.10)$$

substituting the values of X(t) from 6.6 and X_j from 6.7 we get

$$X'(t) = S_i(t) + W(t) - \sum_{j=1}^{N} (S_{ij} + W_j)\phi_j(t)$$
$$= W(t) - \sum_{j=1}^{N} W_j \phi_j(t)$$
$$= W'(t)$$

which depends only on noise W(t) at the front end of the receiver and not at all on the transmitted signal $s_i(t)$. Thus we may express the received random process as

$$X(t) = \sum_{j=1}^{N} X_j \phi_j(t) + X'(t)$$
$$= \sum_{j=1}^{N} X_j \phi_j(t) + W'(t)$$

Now we may characterize the set of correlator output, $\{X_j\}$, j = 1 to N , since the received random process X(t) is Gaussian , we deduce that each X_j is a Gaussian random variable. Hence, each Xj is characterized completely by its mean and variance.

Mean and variance:

The noise process W(t) has zero mean, hence the random variable W_j extracted from W(t) also has zero mean. Thus the mean value of he j^{th} correlator output depends only on S_{ij} as

$$m_{xj} = E[X_j] \ldots\ldots\ldots\ldots\ldots\ldots \textit{from} \quad \textit{eqn}\ 6.7$$
$$= E[S_{ij} + W_j]$$
$$= S_{ij} + E[W_j] \quad \textit{but} \quad E[W_j] = 0$$
$$= S_{ij}$$

variance of X_j is given by

$$\sigma^2{}_{xj} = Var[X_j]$$
$$= E[(X_j - m_{xj})^2] \quad \textit{substituting} \quad m_{xj} = S_{ij}$$
$$= E[(X_j - S_{ij})^2] \quad \textit{from} \quad \textit{equton} \quad 6.7$$
$$= E[W_j^2]$$

substituting the value of W_j from eqn 6.9

$$\sigma^2{}_{xj} = E\left[\int_0^T W(t)\phi_j(t)\,dt \int_0^T W(u)\phi_j(u)\,du\right]$$

$$= E\left[\int_0^T\int_0^T \phi_j(t)\phi_j(u)W(t)W(u)\ dt\,du\right]$$

$$\sigma^2{}_{xj} = \int_0^T\int_0^T \phi_j(t)\phi_j(u)E[W(t)W(u)]\ dt\,du$$

$$= \int_0^T\int_0^T \phi_j(t)\phi_j(u)R_w(t,u)\,dt\,du \cdots\cdots\cdots(6.11)$$

where

$R_w(t,u)$ = $E[W(t)W(u)]$ autocorrelation function of the noise process W(t).Science the noise is stationary, with psd $N_0/2$,$R_w(t,u)$ depends only on the time difference (t-u) and expressed as

$$R_w(t,u) = \frac{N_0}{2}\delta(t-u)\cdots\cdots\cdots(6.12)$$

substituting this value in the equation 6.11 we get

$$\sigma^2{}_{xj} = \frac{N_0}{2}\int_0^T\int_0^T \phi_j(t)\phi_j(u)\delta(t-u)\,dt\,du$$

$$= \frac{N_0}{2}\int_0^T \phi_j^2(t)\,dt$$

Science the $\phi_j(t)$ have unit energy, the above equation reduce to

$$\sigma^2{}_{xj} = \frac{N_0}{2} \qquad for\ \ all\ \ \ j$$

This shows that all the correlator outputs $\{X_j\}$, j = 1 to N have a variance equal to the psd $N_o/2$ of the additive noise process W(t).

Science the $\phi_j(t)$ forms an orthogonal set, then the Xj are mutually uncorrelated, as shown by

$$
\begin{aligned}
Cov[X_j X_k] &= E[(X_j - m_{xj})(X_k - m_{xk})] \\
&= E[(X_j - S_{ij})(X_k - S_{ik})] \\
&= E[W_j W_k] \\
&= E\left[\int_0^T W(t)\phi_j(t)dt \int_0^T W(u)\phi_k(u)du\right] \\
&= \int_0^T \int_0^T \phi_j(t)\phi_k(u)R_w(t,u)dt\,du \\
&= \frac{N_0}{2}\int_0^T \int_0^T \phi_j(t)\phi_k(u)\delta(t-u)dt\,du \\
&= \frac{N_0}{2}\int_0^T \phi_j(t)\phi_k(u)dt \\
&= 0 \qquad j \neq k
\end{aligned}
$$

Since the Xj are Gaussian random variables, from the above equation it is implied that they are also statistically independent.