CHAPTER 5

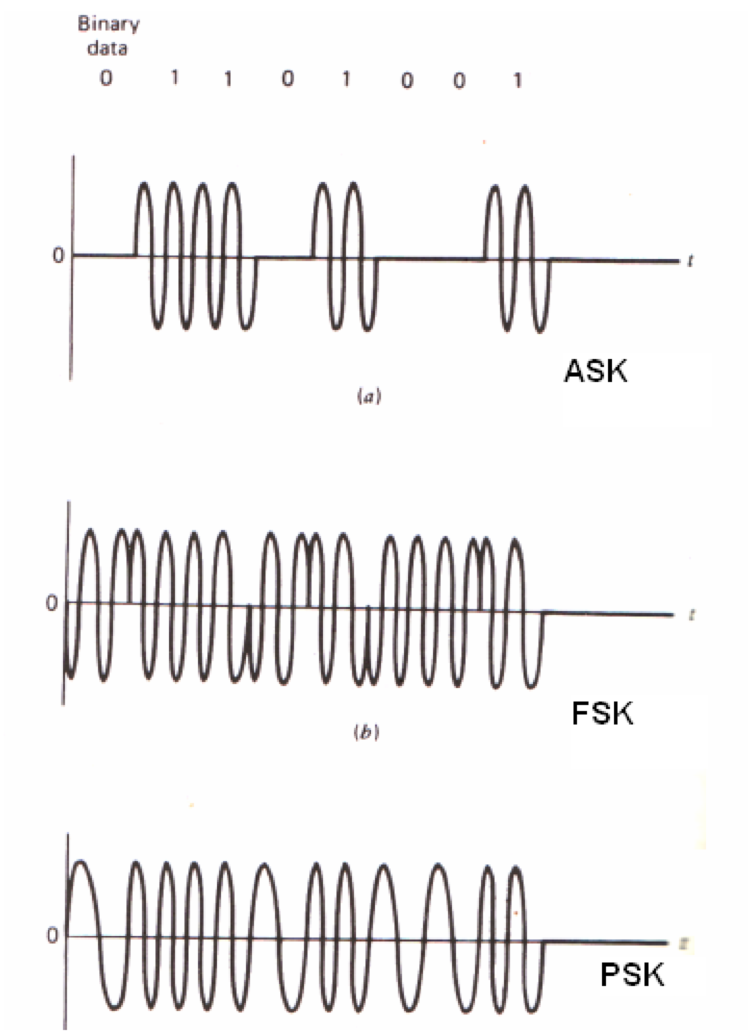
Digital modulation techniques

Modulation is defined as the process by which some characteristics of a carrier is varied in accordance with a modulating wave. In digital communications, the modulating wave consists of binary data or an M-ary encoded version of it and the carrier is sinusoidal wave.

Different Shift keying methods that are used in digital modulation techniques are

- **Amplitude shift keying [ASK]**
- **Frequency shift keying [FSK]**
- **Phase shift keying [PSK]**

Fig shows different modulations



1. ASK[Amplitude Shift Keying]:

In a binary ASK system symbol '1' and '0' are transmitted as

$$S_1(t)=\sqrt{\frac{2E_b}{T_b}}\,Cos2\pi f_1 t \quad \text{for symbol 1}$$

$$S_2(t)=0 \quad \text{for symbol 0}$$

2. FSK[Frequency Shift Keying]:

In a binary FSK system symbol '1' and '0' are transmitted as

$$S_1(t)=\sqrt{\frac{2E_b}{T_b}}\,Cos2\pi f_1 t \quad \text{for symbol 1}$$

$$S_2(t)=\sqrt{\frac{2E_b}{T_b}}\,Cos2\pi f_2 t \quad \text{for symbol 0}$$

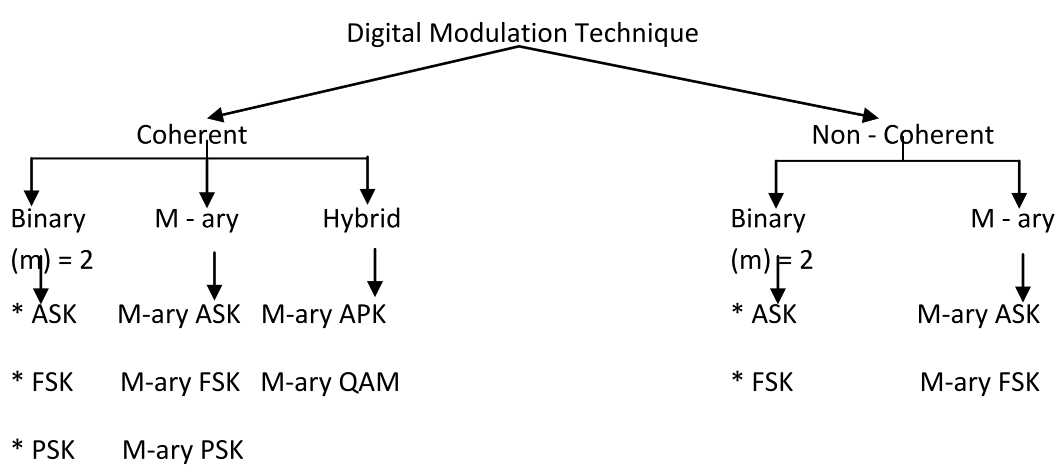
3. PSK[Phase Shift Keying]:

In a binary PSK system the pair of signals $S_1(t)$ and $S_2(t)$ are used to represent binary symbol '1' and '0' respectively.

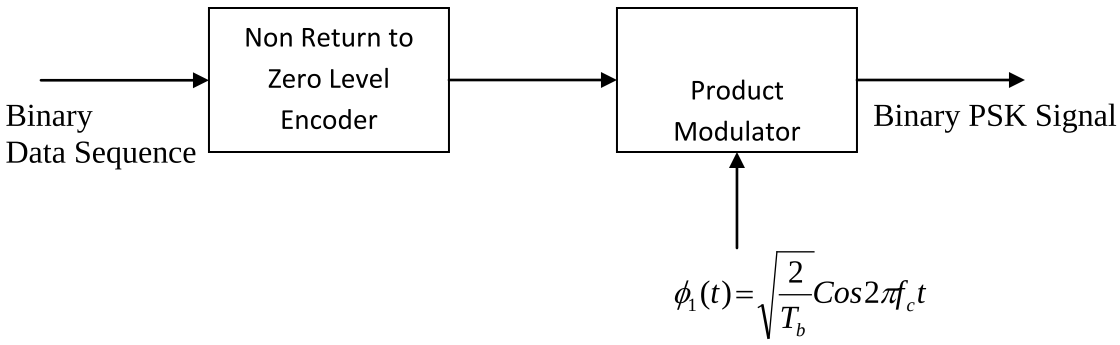
$$S_1(t)=\sqrt{\frac{2E_b}{T_b}}\,Cos2\pi f_c t \quad \text{--------- for Symbol '1'}$$

$$S_2(t)=\sqrt{\frac{2E_b}{T_b}}\,Cos(2\pi f_c t+\pi) = -\sqrt{\frac{2E_b}{T_b}}\,Cos2\pi f_c t \quad \text{------- for Symbol '0'}$$

Hierarchy of digital modulation technique



Coherent Binary PSK:



Fig(a) Block diagram of BPSK transmitter

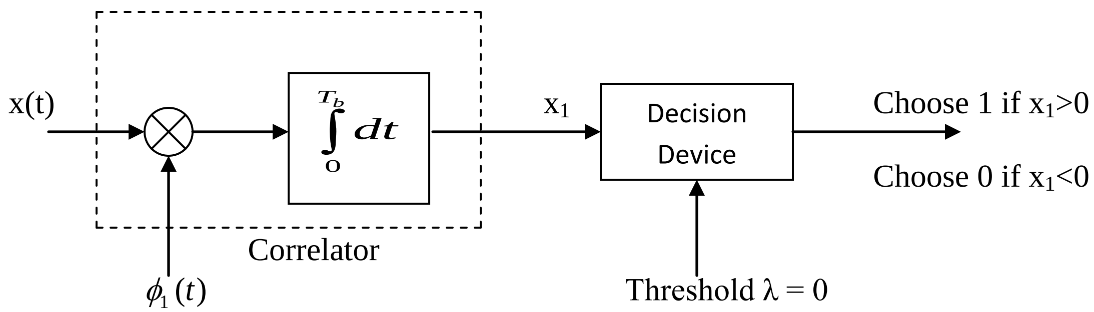


Fig (b) Coherent binary PSK receiver

In a Coherent binary PSK system the pair of signals $S_1(t)$ and $S_2(t)$ are used to represent binary symbol '1' and '0' respectively.

$$S_1(t) = \sqrt{\frac{2E_b}{T_b}} Cos 2\pi f_c t \quad \text{---------} \quad \text{for Symbol '1'}$$

$$S_2(t) = \sqrt{\frac{2E_b}{T_b}} Cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} Cos 2\pi f_c t \quad \text{-------} \text{ for Symbol '0'}$$

Where E_b= Average energy transmitted per bit $E_b = \frac{E_{b0} + E_{b1}}{2}$

In the case of PSK, there is only one basic function of Unit energy which is given by

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} Cos 2\pi f_c t \qquad 0 \le t \le T_b$$

Therefore the transmitted signals are given by

$S_1(t)=\sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b \qquad for \quad Symbol\, 1$

$S_2(t)=-\sqrt{E_b}\phi_1(t) \qquad 0 \le t \le T_b \qquad for \quad Symbol\, 0$

A Coherent BPSK is characterized by having a signal space that is one dimensional (N=1) with two message points (M=2)

$$S_{11} = \int_0^{T_b} S_1(t)\phi_1(t)\,dt = +\sqrt{E_b}$$

$$S_{21} = \int_0^{T_b} S_2(t)\phi_1(t)\,dt = -\sqrt{E_b}$$

The message point corresponding to S_1(t) is located at $S_{11} = +\sqrt{E_b}$ and S_2(t) is located at $S_{21} = -\sqrt{E_b}$.

To generate a binary PSK signal we have to represent the input binary sequence in polar form with symbol '1' and '0' represented by constant amplitude levels of $+\sqrt{E_b}\ \&-\sqrt{E_b}$ respectively. This signal transmission encoding is performed by a NRZ level encoder. The resulting binary wave [in polar form] and a sinusoidal carrier $\phi_1(t)$ [whose frequency $f_c = \frac{n_c}{T_b}$] are applied to a product modulator. The desired BPSK wave is obtained at the modulator output.

To detect the original binary sequence of 1's and 0's we apply the noisy PSK signal x(t) to a Correlator, which is also supplied with a locally generated coherent reference signal $\phi_1(t)$ as shown in fig (b). The correlator output x_1 is compared with a threshold of zero volt.

If $x_1 > 0$, the receiver decides in favour of symbol 1.

If $x_1 < 0$, the receiver decides in favour of symbol 0.

Probability of Error Calculation 'Or' Bit Error rate Calculation [BER Calculation] :-

In BPSK system the basic function is given by

$$\phi_1(t)=\sqrt{\frac{2}{T_b}}\,Cos 2\pi f_c t \qquad 0 \le t \le T_b$$

The signals S_1(t) and S_2(t) are given by

$$S_1(t)=\sqrt{E_b}\,\phi_1(t) \qquad 0 \le t \le T_b \qquad for \;\; Symbol\, 1$$

$$S_2(t)=-\sqrt{E_b}\,\phi_1(t) \qquad 0 \le t \le T_b \qquad for \;\; Symbol\, 0$$

The signal space representation is as shown in fig (N=1 & M=2)

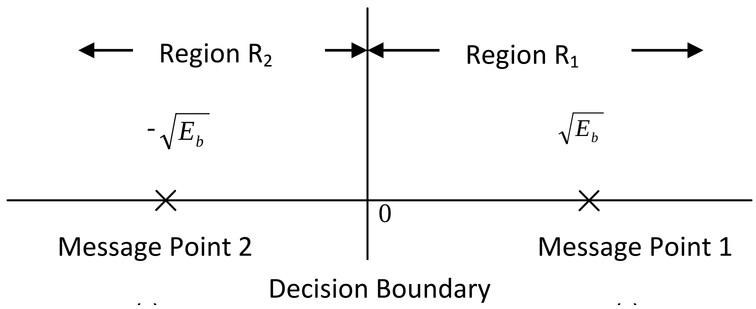


Fig. Signal Space Representation of BPSK

The observation vector x_1 is related to the received signal x(t) by

$$x_1 = \int_0^T x(t)\phi_1(t)\,dt$$

If the observation element falls in the region R_1, a decision will be made in favour of symbol '1'. If it falls in region R_2 a decision will be made in favour of symbol '0'.

The error is of two types

1) $P_e(0/1)$ i.e. transmitted as '1' but received as '0' and

2) $P_e(1/0)$ i.e. transmitted as '0' but received as '1'.

Error of 1st kind is given by

$$P_e(1/0)=\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}\exp\left[\frac{(x_1-\mu)^2}{2\sigma^2}\right]dx_1 \quad \text{Assuming Gaussian Distribution}$$

Where μ = mean value = $-\sqrt{E_b}$ for the transmission of symbol '0'

σ^2 = Variance = $\frac{N_0}{2}$ for additive white Gaussiance noise.

Threshold Value λ = 0. [Indicates lower limit in integration]

Therefore the above equation becomes

$$P_{e0} = P_e(1/0) = \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left[-\frac{(x_1 + \sqrt{E_b})^2}{N_0}\right] dx_1$$

Put $Z = \frac{x_1 + \sqrt{E_b}}{\sqrt{N_0}}$

$$P_{e0} = P_e(1/0) = \frac{1}{\sqrt{\pi}} \int_{\sqrt{(E_b/N_0)}}^{\infty} \exp\left[(-Z)^2\right] dz$$

$$P_e(1/0) = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$

Similarly $P_e(0/1) = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$

The total probability of error $P_e = P_e(1/0)\, P_e(0) + P_e(0/1)\, P_e(1)$ assuming probability of 1's and 0's are equal.

$$P_e = \frac{1}{2}[P_e(1/0) + P_e(0/1)]$$

$$P_e = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$

Coherent Binary FSK

In a binary FSK system symbol '1' and '0' are transmitted as

$S_1(t) = \sqrt{\frac{2E_b}{T_b}} Cos 2\pi f_1 t$ for symbol 1

$S_2(t) = \sqrt{\frac{2E_b}{T_b}} Cos 2\pi f_2 t$ for symbol 0

Frequency $f_i = \frac{n_c + i}{T_b}$ for some fixed integer n_c and i=1, 2

The basic functions are given by

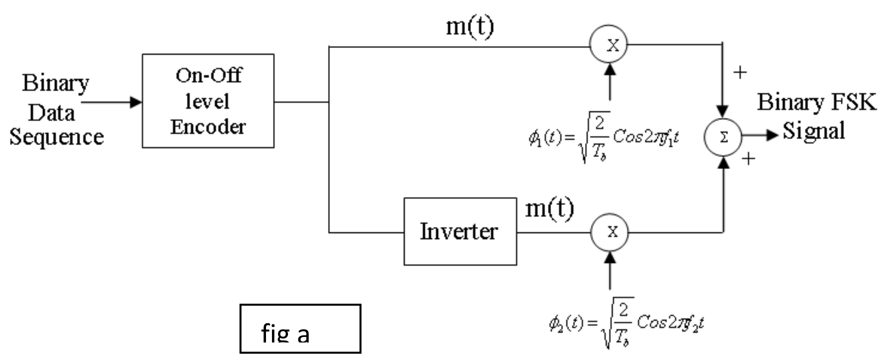
$\phi_1(t) = \sqrt{\frac{2}{T_b}} Cos 2\pi f_1 t$ and

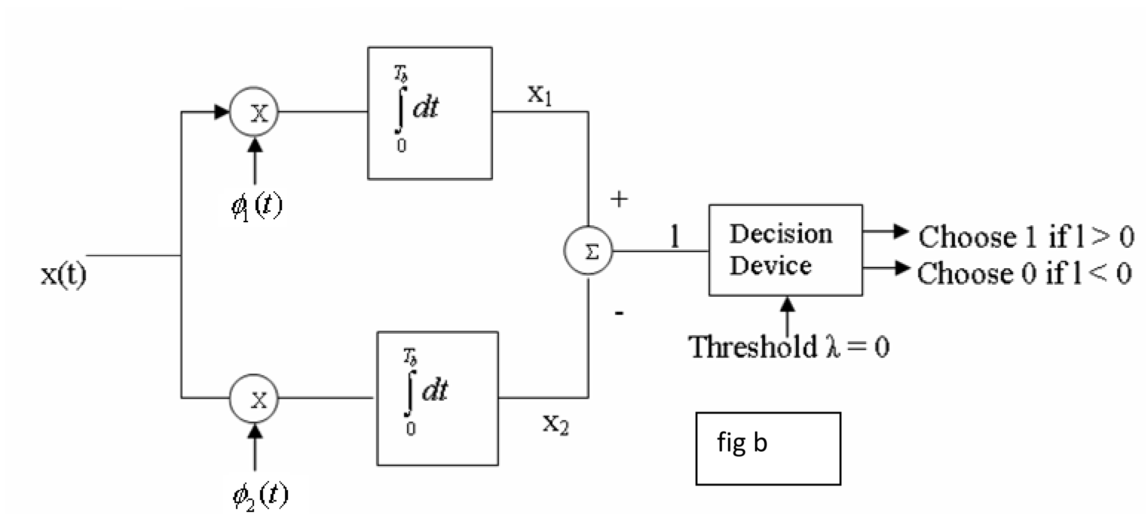
$\phi_2(t) = \sqrt{\frac{2}{T_b}} Cos 2\pi f_2 t \quad for \quad 0 \le t \le T_b \quad and \quad Zero\ Otherwise$

Therefore FSK is characterized by two dimensional signal space with two message points i.e. N=2 and m=2.
The two message points are defined by the signal vector

$$S_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

<u>Generation and Detection:-</u>





<u>fig:</u> FSK transmitter and receiver

A binary FSK Transmitter is as shown in fig. (a). The incoming binary data sequence is applied to on-off level encoder. The output of encoder is $\sqrt{E_b}$ volts for symbol 1 and 0 volts for symbol '0'. When we have symbol 1 the upper channel is switched on with oscillator frequency f_1, for symbol '0', because of inverter the lower channel is switched on with oscillator frequency f_2. These two frequencies are combined using an adder circuit and then transmitted. The transmitted signal is nothing but required BFSK signal.

The detector consists of two correlators. The incoming noisy BFSK signal x(t) is common to both correlator. The Coherent reference signal $\phi_1(t)$ *and* $\phi_2(t)$ are supplied to upper and lower correlators respectively.

The correlator outputs are then subtracted one from the other and resulting a random vector 'l' ($l = x_1 - x_2$). The output 'l' is compared with threshold of zero volts.

If $l > 0$, the receiver decides in favour of symbol 1.

$l < 0$, the receiver decides in favour of symbol 0.

Probability of Error Calculation:-

In binary FSK system the basic functions are given by

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} Cos 2\pi f_1 t \qquad 0 \le t \le T_b$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} Cos 2\pi f_2 t \qquad 0 \le t \le T_b$$

The transmitted signals $S_1(t)$ and $S_2(t)$ are given by

$$S_1(t) = \sqrt{E_b}\,\phi_1(t) \quad \text{for symbol 1}$$

$$S_2(t) = \sqrt{E_b}\,\phi_2(t) \quad \text{for symbol 0}$$

Therefore Binary FSK system has 2 dimensional signal space with two messages $S_1(t)$ and $S_2(t)$, [N=2 , m=2] they are represented as shown in fig.

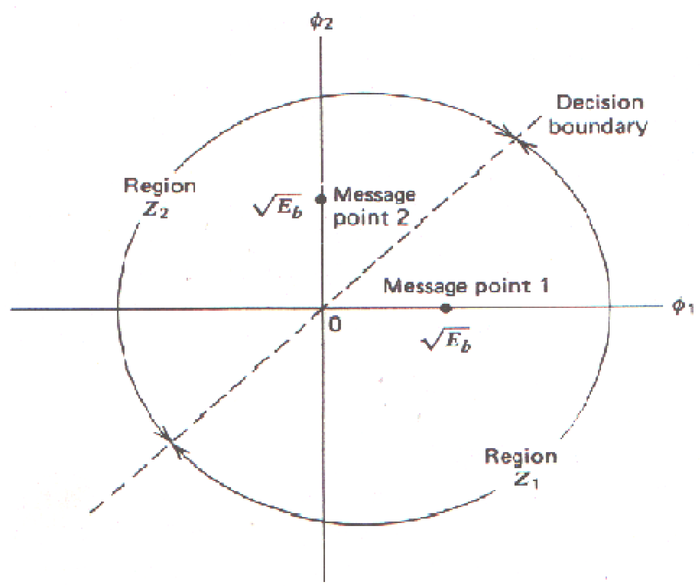


Fig. Signal Space diagram of Coherent binary FSK system.

The two message points are defined by the signal vector

$$S_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

The observation vector x_1 and x_2 (output of upper and lower correlator) are related to input signal x(t) as

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)dt \quad \text{and}$$

$$x_2 = \int_0^{T_b} x(t)\phi_2(t)dt$$

Assuming zero mean additive white Gaussian noise with input PSD $\frac{N_0}{2}$. with variance $\frac{N_0}{2}$.

The new observation vector 'l' is the difference of two random variables x_1 & x_2.

$l = x_1 - x_2$

When symbol '1' was transmitted x_1 and x_2 has mean value of 0 and $\sqrt{E_b}$ respectively.

Therefore the conditional mean of random variable 'l' for symbol 1 was transmitted is

$$E\left[\frac{l}{1}\right]=E\left[\frac{x_1}{1}\right]-E\left[\frac{x_2}{1}\right]$$
$$=\sqrt{E_b}-0$$
$$=\sqrt{E_b}$$

Similarly for '0' transmission $E\left[\frac{l}{0}\right]=-\sqrt{E_b}$

The total variance of random variable 'l' is given by

$$Var[l]=Var[x_1]+Var[x_2]$$
$$=N_0$$

The probability of error is given by

$$P_e(1/0)=P_{e0}=\frac{1}{\sqrt{2\pi N_0}}\int_0^{\infty}\exp\left[-\frac{(l+\sqrt{E_b})^2}{2N_0}\right]dl$$

Put $Z=\dfrac{l+\sqrt{E_b}}{\sqrt{2N_0}}$

$$P_{e0}=\frac{1}{\pi}\int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty}\exp(-z^2)dz$$
$$=\frac{1}{2}erfc\left[\sqrt{\frac{E_b}{2N_0}}\right]$$

Similarly $P_{e1}=\dfrac{1}{2}erfc\left[\sqrt{\dfrac{E_b}{2N_0}}\right]$

The total probability of error = $P_e=\dfrac{1}{2}[P_e(1/0)+P_e(0/1)]$

Assuming 1's & 0's with equal probabilities

$$P_e = \frac{1}{2}[P_{e0} + P_{e1}]$$

$$P_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{2N_0}}\right]$$

BINARY ASK SYSTEM:-

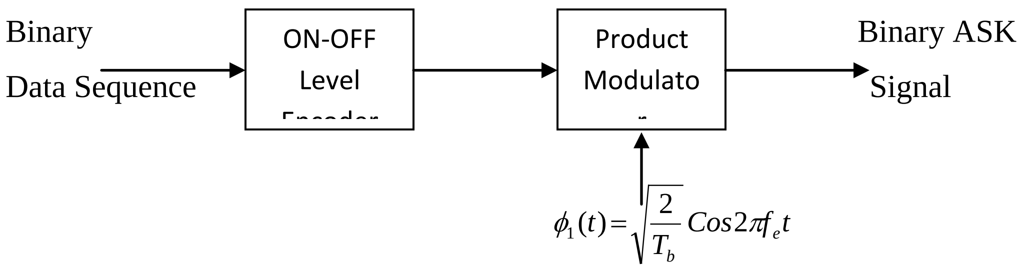


Fig (a) BASK transmitter

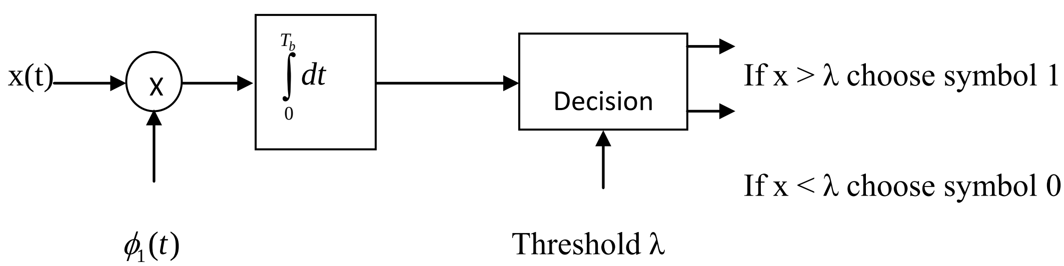


Fig (b) Coherent binary ASK demodulator

In Coherent binary ASK system the basic function is given by

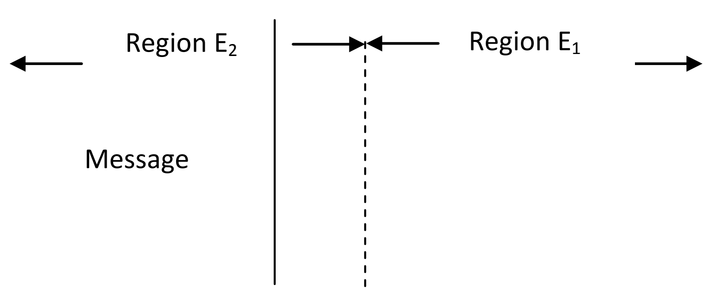
$$\phi_1(t) = \sqrt{\frac{2}{T_b}} Cos 2\pi f_e t \qquad 0 \le t \le T_b$$

The transmitted signals $S_1(t)$ and $S_2(t)$ are given by

$$S_1(t) = \sqrt{E_b}\phi_1(t) \quad for \quad Symbol\ 1$$

$$S_2(t) = 0 \quad for \quad Symbol\ 0$$

The BASK system has one dimensional signal space with two messages (N=1, M=2)



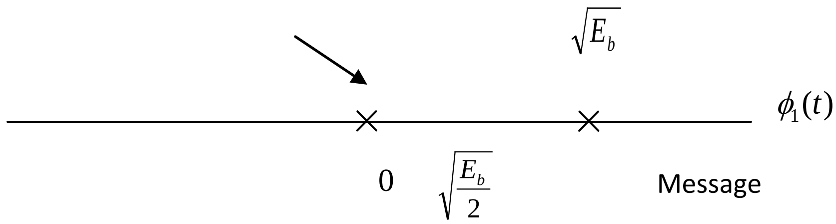


Fig. (c) Signal Space representation of BASK signal

In transmitter the binary data sequence is given to an on-off encoder. Which gives an output $\sqrt{E_b}$ volts for symbol 1 and 0 volt for symbol 0. The resulting binary wave [in unipolar form] and sinusoidal carrier $\phi_1(t)$ are applied to a product modulator. The desired BASK wave is obtained at the modulator output.

In demodulator, the received noisy BASK signal x(t) is apply to correlator with coherent reference signal $\phi_1(t)$ as shown in fig. (b). The correlator output x is compared with threshold λ.

If x > λ the receiver decides in favour of symbol 1.

If x < λ the receiver decides in favour of symbol 0.

BER Calculation:

In binary ASK system the basic function is given by

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} Cos 2\pi f_c t \qquad 0 \le t \le T_b$$

The transmitted signals are given by

$$S_1(t) = \sqrt{E_b}\,\phi_1(t) \quad for \;\; Symbol\ 1$$

$$S_2(t) = 0 \quad for \;\; Symbol\ 0$$

Therefore the average transmitted energy per bit $E_b = \frac{E_{b0} + E_{b1}}{2} = \frac{0 + \frac{A^2 T_b}{2}}{2} = \frac{A^2 T_b}{4}$

The probability of error is given by

$$P_{e0} = \frac{1}{\sqrt{\pi N_0}} \int_{\sqrt{\frac{E_b}{2}}}^{\infty} \exp\left[-\frac{(x-0)^2}{N_o} \right] dx$$

Where ‘x’ is the observed random vector. μ = 0, because the average value for symbol ‘0’ transmission is zero (0).

$\sigma^2 = \frac{N_0}{2}$ assuming additive white Gaussian noise with into PSD $\frac{N_0}{2}$

Let $Z = \frac{x}{\sqrt{N_0}}$

$$P_{e0} = \frac{1}{\pi} \int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty} \exp(-z^2) dz$$

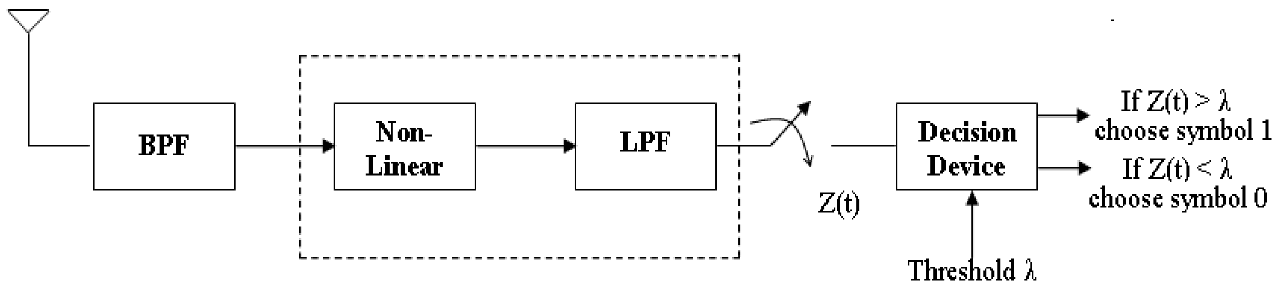
$$= \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{2N_0}}\right]$$

similarly $P_{e1} = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{2N_0}}\right]$

The total probability of error = $\frac{1}{2}[P_{e0} + P_{e1}]$

$$P_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{2N_0}}\right]$$

Incoherent detection:



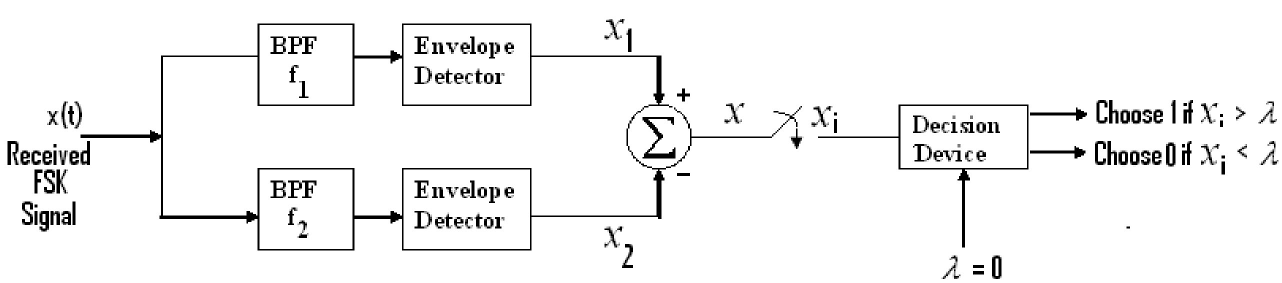
Fig(a). : Envelope detector for OOK BASK

Incoherent detection as used in analog communication does not require carrier for reconstruction. The simplest form of incoherent detector is the envelope detector as shown in figure(a). The output of envelope detector is the baseband signal. Once the baseband signal is recovered, its samples are taken at regular intervals and compared with threshold.

If Z(t) is greater than threshold (λ) a decision will be made in favour of symbol ‘1’

If Z(t) the sampled value is less than threshold (λ) a decision will be made in favour of symbol ‘0’.

<u>**Non- Coherenent FSK Demodulation**</u>:-



Fig(b) : Incoherent detection of FSK

Fig(b) shows the block diagram of incoherent type FSK demodulator. The detector consists of two band pass filters one tuned to each of the two frequencies used to communicate ‘0’s and ‘1’s., The output of filter is envelope detected and then baseband detected using an integrate and dump operation. The detector is simply evaluating which of two possible sinusoids is stronger at the receiver. If we take the difference of the outputs of the two envelope detectors the result is bipolar baseband.

The resulting envelope detector outputs are sampled at $t = kT_b$ and their values are compared with the threshold and a decision will be made infavour of symbol 1 or 0.

<u>Differential Phase Shift Keying:- [DPSK]</u>

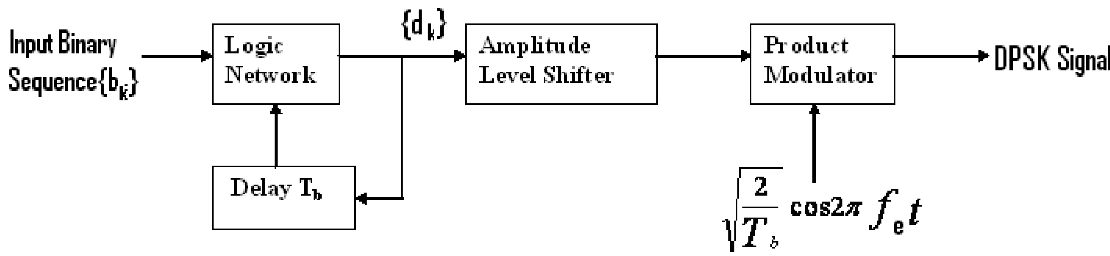


Fig. (a) DPSK Transmitter

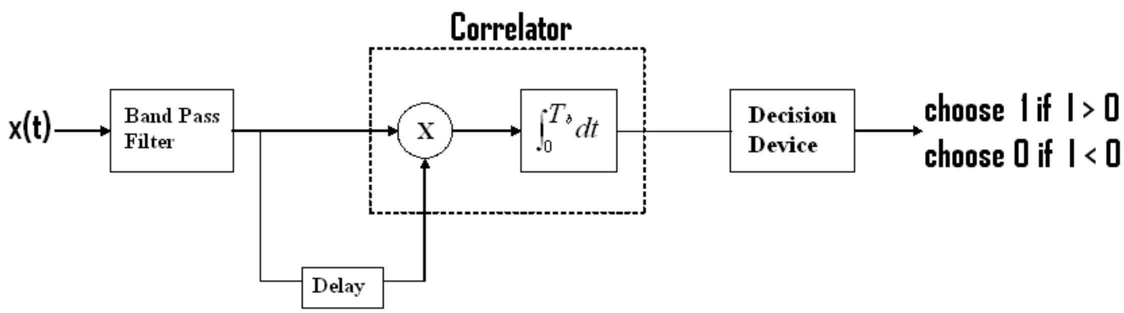


Fig. (b) DPSK Receiver

A DPSK system may be viewed as the non coherent version of the PSK. It eliminates the need for coherent reference signal at the receiver by combining two basic operations at the transmitter

(1) Differential encoding of the input binary wave and

(2) Phase shift keying

Hence the name differential phase shift keying [DPSK]. To send symbol ‘0’ we phase advance the current signal waveform by 180^0 and to send symbol 1 we leave the phase of the current signal waveform unchanged.

The differential encoding process at the transmitter input starts with an arbitrary first but, securing as reference and thereafter the differentially encoded sequence$\{d_k\}$ is generated by using the logical equation.

$$d_k = d_{k-1} b_k \oplus \overline{d_{k-1}}\, \overline{b_k}$$

Where b_k is the input binary digit at time kT_b and d_{k-1} is the previous value of the differentially encoded digit. Table illustrate the logical operation involved in the generation of DPSK signal.

Input Binary Sequence {b_K}	1	0	0	1	0	0	1	1
Differentially Encoded sequence {d_K} 1	1	0	1	1	0	1	1	1
Transmitted Phase 0	0	Π	0	0	Π	0	0	0
Received Sequence (Demodulated Sequence)	1	0	0	1	0	0	1	1

A DPSK demodulator is as shown in fig(b). The received signal is first passed through a BPF centered at carrier frequency f_c to limit noise power. The filter output and its delay version are applied to correlator the resulting output of correlator is proportional to the cosine of the difference between the carrier phase angles in the two correlator inputs. The correlator output is finally compared with threshold of ‘0’ volts .

If correlator output is +ve -- A decision is made in favour of symbol ‘1’

If correlator output is -ve --- A decision is made in favour of symbol ‘0’

COHERENT QUADRIPHASE – SHIFT KEYING

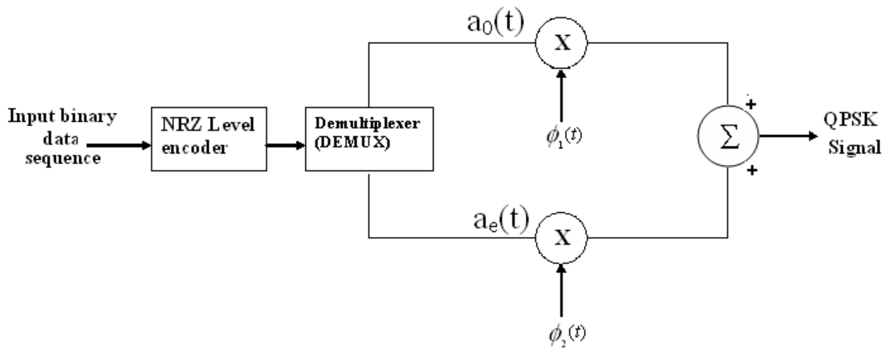


Fig. (a) QPSK Transmitter

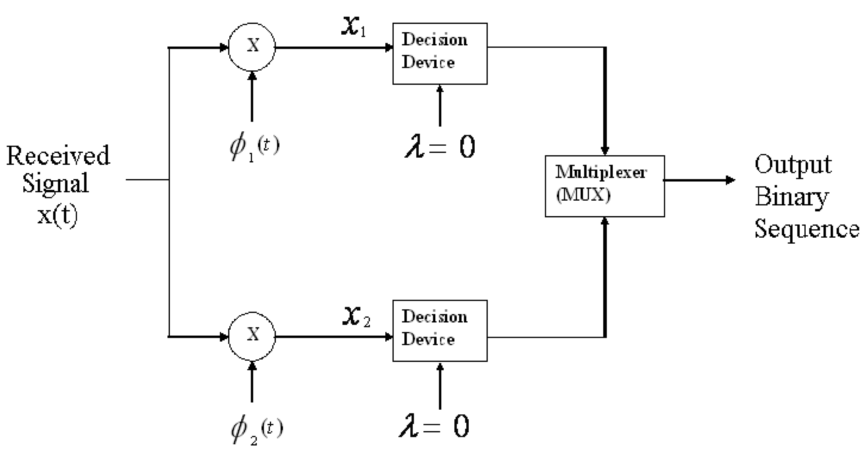


Fig. (b) QPSK Receiver

In case of QPSK the carrier is given by

$$s_i(t) = \sqrt{\frac{2E}{T}}\, Cos[2\pi f_c t + (2i-1)\pi/4] \qquad 0 \le t \le T \quad i = 1 \; to \; 4$$

$$s_i(t) = \sqrt{\frac{2E}{T}}\, Cos[(2i-1)\pi/4]\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\, \sin[(2i-1)\pi/4]\sin(2\pi f_c t) \; 0 \le t \le T \quad i = 1 \; to \; 4$$

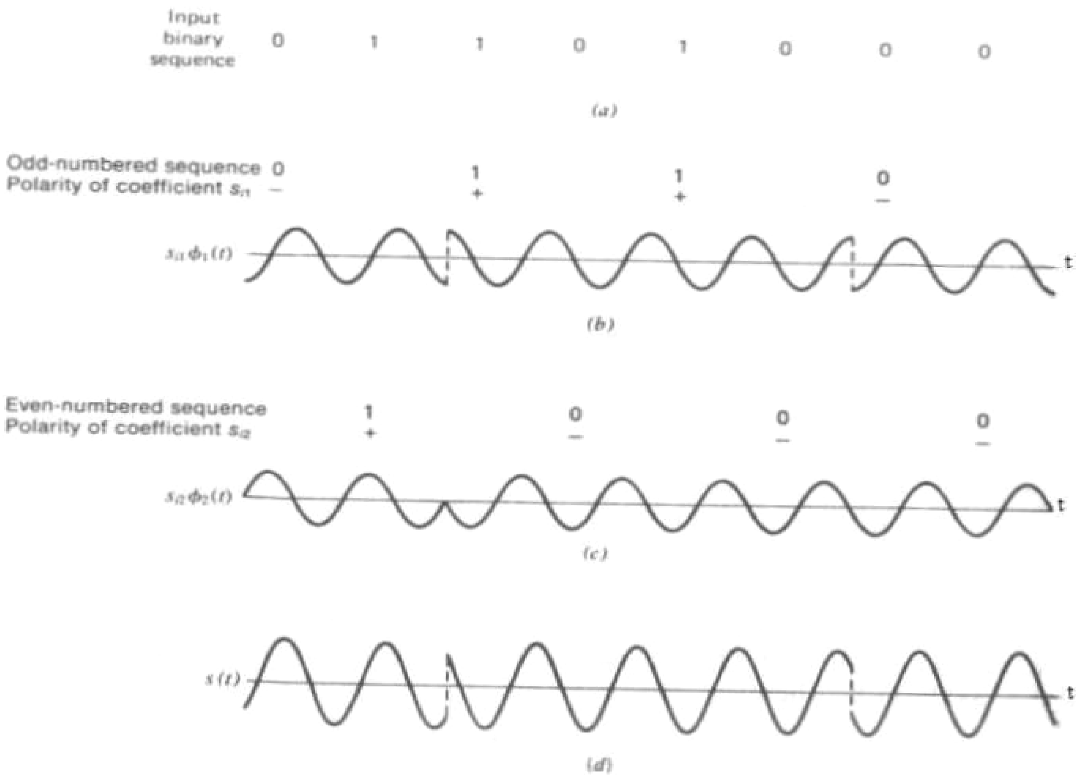


Fig. (c) QPSK Waveform

In QPSK system the information carried by the transmitted signal is contained in the phase. The transmitted signals are given by

$$S_1(t) = \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + \frac{\pi}{4}\right] \quad --- \quad for\, input \quad di\, bit \quad 10$$

$$S_2(t) = \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + \frac{3\pi}{4}\right] \quad --- \quad for\, input \quad dibit \quad 00$$

$$S_3(t) = \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + \frac{5\pi}{4}\right] \quad --- \quad for\, input \quad dibit \quad 01$$

$$S_t(t) = \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + \frac{7\pi}{4}\right] \quad --- \quad for\, input \quad dibit \quad 11$$

Where the carrier frequency $f_C = \frac{n_C}{7}$ for some fixed integer n_c

E = the transmitted signal energy per symbol.

T = Symbol duration.

The basic functions $\phi_1(t)$ *and* $\phi_2(t)$ *are given by*

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos\left[2\pi f_c t\right] \qquad 0 \le t < T$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \sin\left[2\pi f_c t\right] \qquad 0 \le t < T$$

There are four message points and the associated signal vectors are defined by

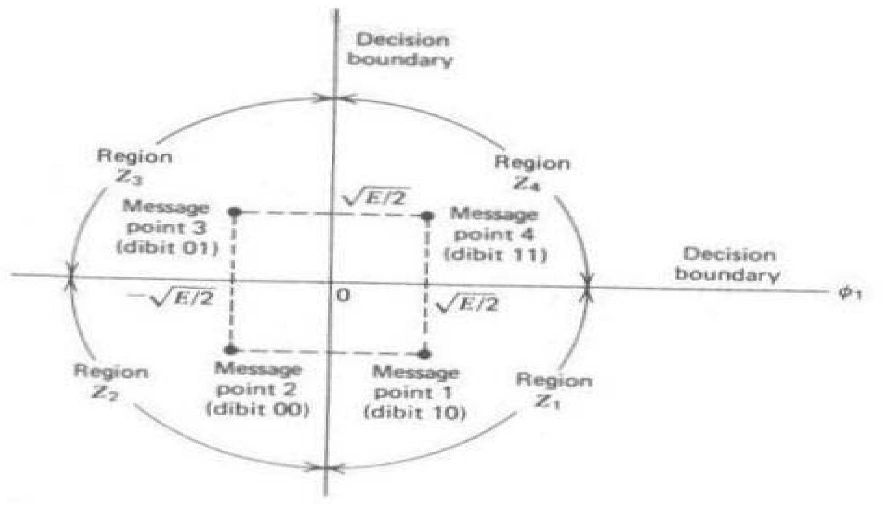
$$Si = \begin{bmatrix} \sqrt{E} \ \cos\left[(2i-1)\frac{\pi}{4}\right] \\ -\sqrt{E} \ \sin\left[(2i-1)\frac{\pi}{4}\right] \end{bmatrix} \qquad i = 1,2,3,4$$

The table shows the elements of signal vectors, namely S_{i1} & S_{i2}

Table:-

Input dibit	Phase of QPSK signal(radians)	Coordinates of message points	
		S_{i1}	S_{i2}
10	$\frac{\pi}{4}$	$+\sqrt{E/2}$	$-\sqrt{E/2}$
00	$\frac{3\pi}{4}$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
01	$\frac{5\pi}{4}$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
11	$\frac{7\pi}{4}$	$+\sqrt{E/2}$	$+\sqrt{E/2}$

Therefore a QPSK signal is characterized by having a two dimensional signal constellation(i.e.N=2)and four message points(i.e. M=4) as illustrated in fig(d)



.Fig (d) Signal-space diagram of coherent QPSK system.

Generation:-

Fig(a) shows a block diagram of a typical QPSK transmitter, the incoming binary data sequence is first transformed into polar form by a NRZ level encoder. Thus the symbols 1& 0 are represented by $+\sqrt{E_b}$ *and* $-\sqrt{E_b}$ respectively. This binary wave is next divided by means of a demultiplexer [Serial to parallel conversion] into two separate binary waves consisting of the odd and even numbered input bits. These two binary waves are denoted by $a_o(t)$ and $a_e(t)$

The two binary waves $a_o(t)$ and $a_e(t)$ are used to modulate a pair of quadrature carriers or orthonormal basis functions $\phi_1(t)$ & $\phi_2(t)$ which are given by

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t$$

&

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin 2\pi f_c t$$

The result is a pair of binary PSK signals, which may be detected independently due to the orthogonality of $\phi_1(t)$ & $\phi_2(t)$.

Finally the two binary PSK signals are added to produce the desired QPSK signal.

Detection:-

The QPSK receiver consists of a pair of correlators with a common input and supplied with a locally generated pair of coherent reference signals $\phi_1(t)$ & $\phi_2(t)$ as shown in fig(b).The correlator outputs x_1 and x_2 produced in response to the received signal x(t) are each compared with a threshold value of zero.

The in-phase channel output :

If $x_1 > 0$ a decision is made in favour of symbol 1

$x_1 < 0$ a decision is made in favour of symbol 0

Similarly quadrature channel output:

If $x_2 > 0$ a decision is made in favour of symbol 1 and

$x_2 < 0$ a decision is made in favour of symbol 0

Finally these two binary sequences at the in phase and quadrature channel outputs are combined in a multiplexer (Parallel to Serial) to reproduce the original binary sequence.

Probability of error:-

A QPSK system is in fact equivalent to two coherent binary PSK systems working in parallel and using carriers that are in-phase and quadrature.

The in-phase channel output x_1 and the Q-channel output x_2 may be viewed as the individual outputs of the two coherent binary PSK systems. Thus the two binary PSK systems may be characterized as follows.

- The signal energy per bit $\sqrt{E/2}$
- The noise spectral density is $\frac{N_0}{2}$

The average probability of bit error in each channel of the coherent QPSK system is

$$P^1 = \frac{1}{2}\ erfc\left[\sqrt{\frac{E/2}{N_0}}\right] \qquad \left(E = E/2\right)$$

$$= \frac{1}{2}\ erfc\left[\sqrt{\frac{E}{2N_0}}\right]$$

The bit errors in the I-channel and Q-channel of the QPSK system are statistically independent . The I-channel makes a decision on one of the two bits constituting a symbol (d_i bit) of the QPSK signal and the Q-channel takes care of the other bit.

Therefore, the average probability of a direct decision resulting from the combined action of the two channels working together is

p_c= probability of correct reception

p^1= probability of error

$$P_C = \left[1 - P^1\right]^2$$

$$= \left[1 - \frac{1}{2} erfc\left[\sqrt{\frac{E}{2No}}\right]\right]^2$$

$$= 1 - erfc\left[\sqrt{\frac{E}{2No}}\right] + \frac{1}{4} erfc^2\left[\sqrt{\frac{E}{2No}}\right]$$

The average probability of symbol error for coherent QPSK is given by

$$P_e = 1 - P_C$$

$$= erfc\left[\sqrt{\frac{E}{2No}}\right] - \frac{1}{4} erfc^2\left[\sqrt{\frac{E}{2No}}\right]$$

In the region where $\frac{E}{2N_o} >> 1$ We may ignore the second term and so the approximate formula for the average probability of symbol error for coherent QPSK system is

$$P_e = erfc\sqrt{\frac{E}{2No}}$$

Minimum shift keying:-

In a continuous phase frequency shift keying [CPFSK] system the transmitted signal is given by

$$S(t) = \begin{cases} \sqrt{\dfrac{2E_b}{T_b}} \cos\left[2\pi f_1 t + \theta(0)\right] & - \quad for \ symbol \ 1 \\ \sqrt{\dfrac{2E_b}{T_b}} \cos\left[2\pi f_2 t + \theta(0)\right] & - \quad for \ symbol \ 0 \end{cases}$$

Where E_b is the transmitted signal energy per bit and T_b is bit duration the CPSK signal S(t) is expressed in the conventional form of an angle modulated signal as

$$S(t) = \sqrt{\frac{2E_b}{T_b}} \cos\left[2\pi f_c t + \theta(0)\right]$$

The phase θ(t) is a continuous function of time which is given by

$$\theta(t) = \theta(0) \pm \frac{\pi h\, t}{T_b} \qquad 0 \le t \le T_b$$

The transmitted frequencies f_1 & f_2 *are given by*

$$f_1 = f_c + \frac{h}{2T_b}$$

$$f_2 = f_c - \frac{h}{2T_b}$$

$$f_c = 1/2(f_1 + f_2)$$

$$h = T_b(f_1 - f_2)$$

Where f_c = *the carrier frequency* &

h = *the deviation ratio*

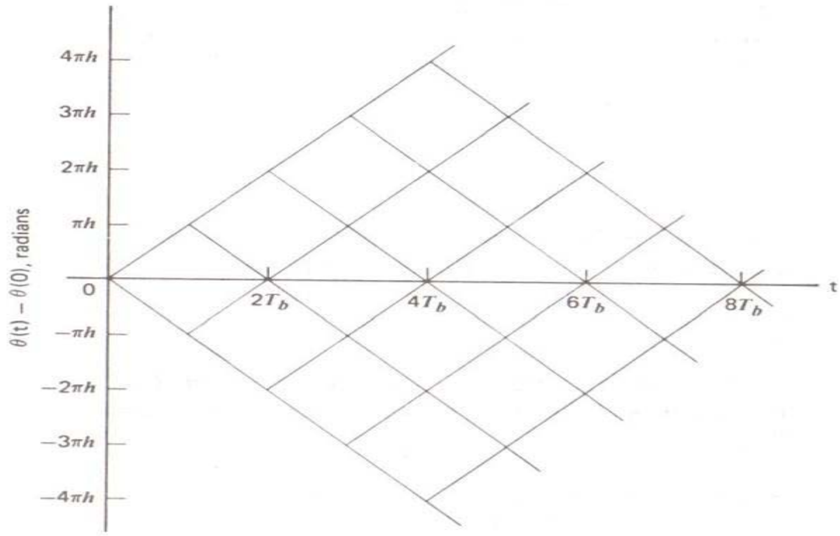
The variation of phase θ(t) with time t follows a path consisting of sequence of straight lines, the slope of which represent frequency change Fig(a) shows the possible paths starting from time t=0. This plot is known as “**Phase tree**”.

When h= ½ the frequency deviation equals half the bit rate. This is the minimum frequency difference (deviation) that allows the two FSK signals representing symbol ‘1’ & ‘0’. For this reason a CPFSK signal with a deviation ratio of one- half is commonly referred to as “**minimum shift keying**”[MSK].

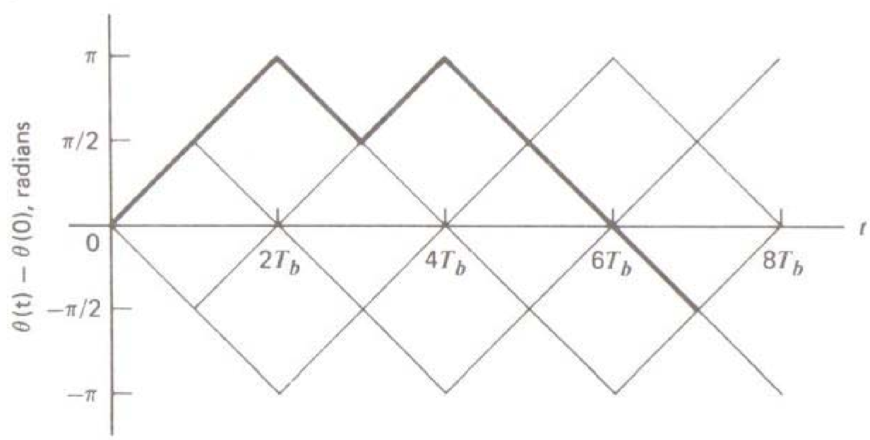
Deviation ratio h is measured with respect to the bit rate $1/T_b$

at $t = T_b$

$$\theta(T_b) - \theta(0) = \begin{cases} \pi h & for\ Symbol\ 1 \\ -\pi h & for\ Symbol\ 0 \end{cases}$$



fig(b) phase tree



Fig(c): Phase Trellis, for sequence 1101000

In terms of In phase and Quadrature Component

$$s(t) = \sqrt{\frac{2E_b}{T_b}} Cos[\theta(t)] Cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} Sin[\theta(t)] Sin(2\pi f_c t)$$

with the deviation ratio h=1/2

$$\theta(t) = \theta(0) \pm \frac{\pi}{2T_b} t \qquad 0 \le t \le T_b$$

+ Sign corresponds to symbol 1

- Sign corresponds to symbol 0

In phase components

For the interval of $-T_b \le t \le T_b$

consists of half cosine pulse

$$\begin{aligned} s_1(t) &= \sqrt{\frac{2E_b}{T_b}} Cos[\theta(t)] \\ &= \sqrt{\frac{2E_b}{T_b}} Cos[\theta(0)]\ Cos\left(\frac{\pi}{2T_b} t\right) \\ &= \pm \sqrt{\frac{2E_b}{T_b}} Cos\left(\frac{\pi}{2T_b} t\right) \qquad -T_b \le t \le T_b \end{aligned}$$

+ Sign corresponds to θ(0) =0

- Sign corresponds to θ(0) = π

Quadrature components

For the interval of $0 \le t \le 2\ T_b$

consists of half sine pulse

$$\begin{aligned} s_Q(t) &= \sqrt{\frac{2E_b}{T_b}}\ Sin[\theta(t)] \\ &= \sqrt{\frac{2E_b}{T_b}}\ Sin[\theta(T_b)]\ Cos\left(\frac{\pi}{2T_b} t\right) \\ &= \pm \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right) \qquad 0 \le t \le 2T_b \end{aligned}$$

+ Sign corresponds to θ(Tb) =π/2

- Sign corresponds to θ(Tb) = -π/2

since the phase states θ(0) and θ(Tb) can each assume one of the two possible values, any one of the four possibilities can arise

1. The Phase $\theta(0) = 0$ and $\theta(T_b) = \pi/2$, corresponding to the transmission of symbol 1.
2. The Phase $\theta(0) = \pi$ and $\theta(T_b) = \pi/2$, corresponding to the transmission of symbol 0.
3. The Phase $\theta(0) = \pi$ and $\theta(T_b) = -\pi/2$ (or, equivalently, $3\pi/2$, modulo 2π), corresponding to the transmission of symbol 1.
4. The Phase $\theta(0) = 0$ and $\theta(T_b) = -\pi/2$, corresponding to the transmission of symbol 0.

Basic functions are given by

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\, Cos\left(\frac{\pi}{2T_b}t\right) Cos\,(2\pi f_c t) \qquad -T_b \le t \le T_b$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}}\, Sin\left(\frac{\pi}{2T_b}t\right) Sin\,(2\pi f_c t) \qquad 0 \le t \le 2T_b$$

we may express the MSK signal in the form

$$s(t) = s_1\phi_1(t) + s_2\phi_2(t) \qquad 0 \le t \le T_b$$

The coefficients are given by

$$s_1 = \int_{-T_b}^{T_b} s(t)\,\phi_1(t)\,dt$$
$$= \sqrt{E_b}\; Cos\left[\theta(0)\right] \qquad -T_b \le t \le T_b$$

and

$$s_2 = \int_{0_b}^{2T_b} s(t)\,\phi_2(t)\,dt$$
$$= -\sqrt{E_b}\; Sin\left[\theta(T_b)\right] \qquad 0 \le t \le 2T_b$$

The signal space diagram for MSK system is as shown in fig

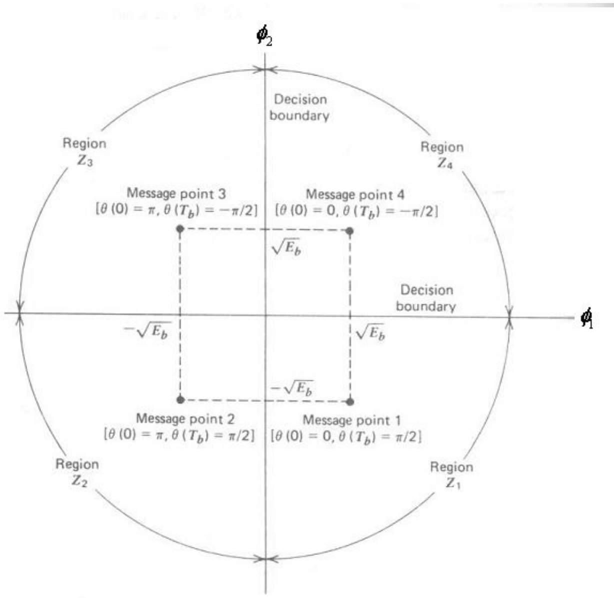


Fig: signal space diagram for MSK system

Signal Space Characterization of MSK

Transmitted binary symbol, $0 \leq t \leq T_b$	Phase states (*radians*)		Coordinates of message points	
	$\theta(0)$	$\theta(T_b)$	s_1	s_2
1	0	$+\pi/2$	$+\sqrt{E_b}$	$-\sqrt{E_b}$
0	π	$+\pi/2$	$-\sqrt{E_b}$	$-\sqrt{E_b}$
1	π	$-\pi/2$	$-\sqrt{E_b}$	$+\sqrt{E_b}$
0	0	$-\pi/2$	$+\sqrt{E_b}$	$+\sqrt{E_b}$

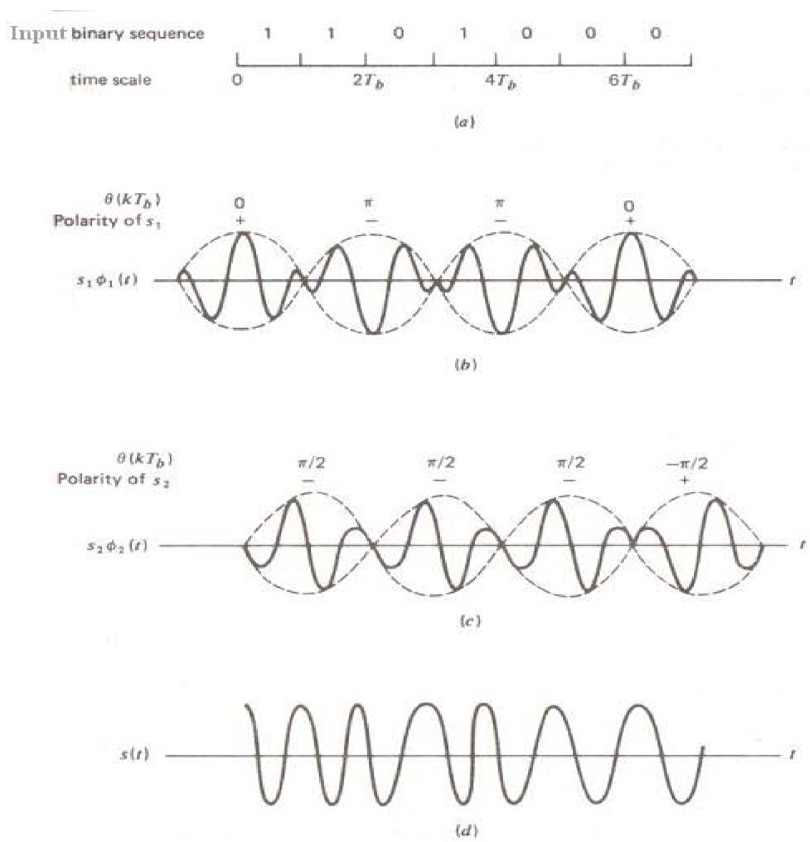


Fig: sequence and waveforms for MSK signal

(a) input binary sequence (b) scaled time function $s_1\phi_1(t)$

(c) scaled time function $s_2\phi_2(t)$ (d)obtained by adding (b) and (c)

In the case of AWGN channel, the received signal is given by

x(t) = s(t) + w(t)

where s(t) is the transmitted MSK signal and w(t) is the sample function of a white Gaussian noise.

The projection of the received signal x(t) onto the reference signal $\phi_1(t)$ is

$$x_1 = \int_{-T_b}^{T_b} x(t)\,\phi_1(t)\,dt$$

$$= s_1 + w_1 \qquad -T_b \le t \le T_b$$

similarly the projection of the received signal x(t) onto the reference signal $\phi_2(t)$ is

$$x_2 = \int_{0}^{2T_b} x(t)\,\phi_2(t)\,dt$$

$$= s_2 + w_2 \qquad 0 \le t \le 2T_b$$

If $x_2 > 0$, the receiver chooses the estimate $\hat{\theta}(T_b) = -\frac{\pi}{2}$. If, on the other hand, x2<0, it chooses the estimate $\hat{\theta}(T_b) = \frac{\pi}{2}$.

To reconstruct the original binary sequence, we interleave the above two sets of phase decisions,

1 If we have the estimates $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = -\frac{\pi}{2}$, or alternatively if we have the estimates $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = \frac{\pi}{2}$, the receiver makes a final decision in favor of symbol 0.

2 If we have the estimates $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = -\frac{\pi}{2}$, or alternatively if we have the estimates $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = \frac{\pi}{2}$, the receiver makes a final decision in favor of symbol 1.

Generation and detection of MSK signal:-

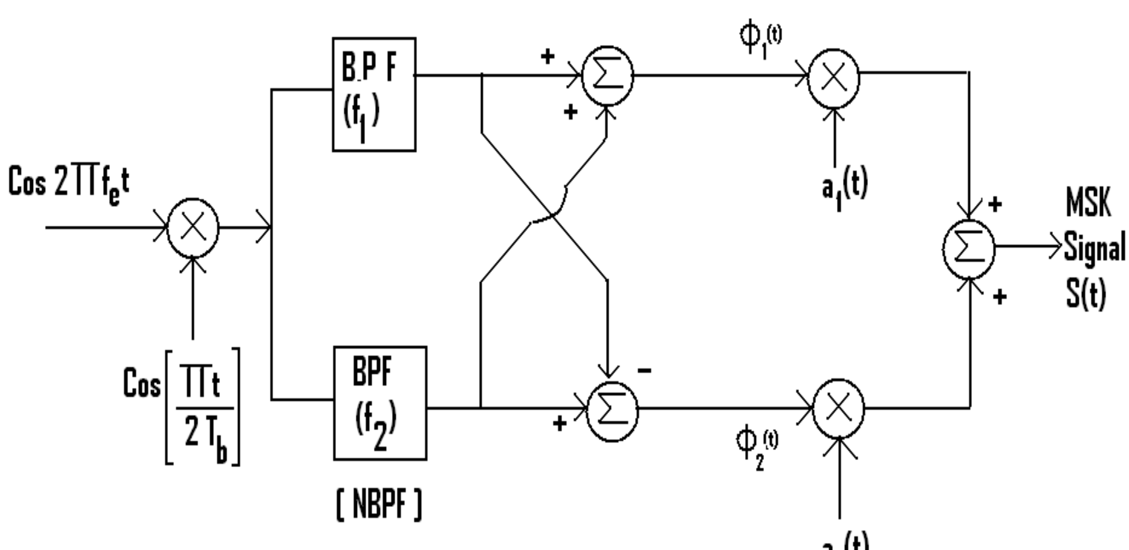


Fig (a) : MSK Transmitter

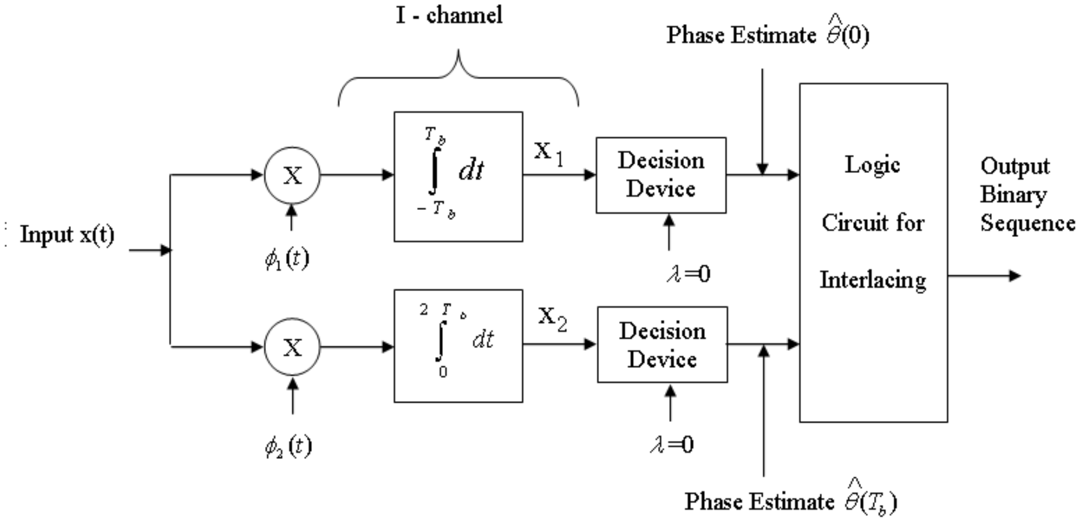


Fig. (b) MSK Receiver

Fig (a) shows the block diagram of typical MSK transmitter. and (b)receiver

Two input sinusoidal waves one of frequency $f_c = \frac{n_C}{4T_b}$ for some fixed integer n_c and the other of frequency $\frac{1}{4T_b}$ are first applied to a modulator. This produces two phase coherent sinusoidal waves at frequencies f_1 and f_2 which are related to the carrier frequency f_c and the bit rate R_b by

$$f_1 = f_c + \frac{h}{2T_b} \quad or \quad f_C + \frac{h}{2}R_b$$

$$f_2 = f_c - \frac{h}{2T_b} \quad or \quad f_C - \frac{h}{2}R_b \qquad for \;\; h = \frac{1}{2}$$

These two sinusoidal waves are separated from each other by two narrow band filters one centered at f_1 and the other at f_2. The resulting filter outputs are next linearly combined to produce the pair of basis functions $\phi_1(t)$ *and* $\phi_2(t)$. Finally $\phi_1(t)$ *and* $\phi_2(t)$ are multiplied with two binary waves $a_1(t)$ and $a_2(t)$ both of which have a bit rate equal to $\frac{1}{2T_b}$. These two binary waves are extracted from the incoming binary sequence.

Fig (b) shows the block diagram of a typical MSK receiver. The received signal x(t) is correlated with locally generated replicas of the coherent reference signals $\phi_1(t)$ *and* $\phi_2(t)$. The integration in the Q – channel is delayed by T_b seconds with respect to the I - channel.

The resulting in-phase and quadrature channel correlator outputs x_1 and x_2 are each compared with a threshold of zero. To estimate the phase $\theta(0)$ and $\theta(T_b)$. Finally these phase decisions are interleaved so as to reconstruct the original input binary sequence with a minimum average probability of symbol error in an AGWN channel.

PROBLEM 1.

Binary data has to be transmitted over a telephone link that has a usable bandwidth of 3000Hz and a maximum achievable signal-to-noise power ratio of 6 dB at its output..

a. Determine the maximum signaling rate and probability of error if a coherent ASK scheme is used for transmitting binary data through this channel.
b. If the data is maintained at 300 bits/sec, calculate the error probability.

Solution:

a) If we assume that an ASK signal requires a bandwidth of $3r_b$ Hz, then the maximum signaling rate permissible is given by

Bandwidth =3 r_b=3000 Hz

r_b=1000 bits/sec.

$$Average\ Signal\ Power = A^2/4$$
$$Noise\ Power = (2)(\eta/2)(3000)$$
$$\frac{Average\ Signal\ Power}{Noise\ Power} = 4 = \frac{A^2}{12{,}000\eta}$$
$$\frac{A^2}{\eta} = 48{,}000$$
$$Hence,\ A^2/4\eta r_b = 12\ \ and$$
$$P_e = Q\left(\sqrt{12}\right) = Q(3.464) \cong 0.0003$$

b) If the bit rate is reduced to 300 bits/sec, then

$$\frac{A^2}{4\eta r_b} = 40$$
$$Pe = Q\left(\sqrt{40}\right) = Q(6.326) \cong 10^{-10}$$

PROBLEM 2

Binary data is transmitted over an RF band pass channel with a usable bandwidth of

10 MHz at a rate of (4.8) (10^6) bits/sec using an ASK signaling method. The carrier amplitude at the receiver antenna is 1 mv and the noise power spectral density at the receiver input is 10^{-15} watt/Hz. Find the error probability of a coherent and non coherent receiver..

Solution:

a) The bit error probability for the coherent demodulator is

$$P_e = Q\left(\sqrt{\frac{A^2 T_b}{4\eta}}\right); \qquad A = 1\ mv, \qquad T_b = 10^{-6}/4.8$$

$$\eta/2 = 10^{-15}\ watt/Hz$$

$$Pe = Q\left(\sqrt{26}\right) \cong 2(10^{-7}).$$

b) for non coherent ASK p_e is given by

$$\left(P_e\right) = \frac{1}{2}\exp\left[-\left(A^2 T_b/(16\eta)\right)\right],$$

$p_e = 0.0008$

PROBLEM 3.

Binary data is transmitted at a rate of 10^6 bits/sec over a microwave link having a bandwidth of 3 MHz. Assume that the noise power spectral density at the receiver input is $\eta/2 = 10^{-10}\ watt/Hz.$ Find the average carrier power required at the receiver input for coherent PSK and DPSK signaling schemes to maintain $P_e \leq 10^{-4}$.

Solution:

The probability of error for the PSK scheme is

$$\left(P_e\right)_{PSK} = Q\left(\sqrt{2S_{av}T_b/\eta}\right) \leq 10^{-4},$$

thus

$$\sqrt{2S_{av}T_b/\eta} \geq 3.75$$

$$(S_{av}) \geq (3.75)^2\,(10^{-10})(10^6) = 1.48dBm$$

For the DPSK scheme

$$\left(P_e\right)_{DPSK} = \frac{1}{2}\exp\left[-\left(A^2T_b/2\eta\right)\right] \leq 10^{-4},$$

Hence,

$$S_{av}\,T_b/\eta \geq 8.517$$

$$\left(S_{av}\right)_{DPSK} \geq 2.3.3\,dBm$$

This example illustrates that the DPSK signaling scheme requires about 1 dB more power than the coherent PSK scheme when the error probability is of the order of 10^{-4}.

Probability of Error

Definition: Defines average probability of error that can occur in a Communication system

Error Functions

(1) Error function erf(u):

$$erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-z^2) dz \qquad \text{-------- (A6.1)}$$

(2) Complementary error function erfc(u):

$$erfc(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2) dz \qquad \text{-------- (A6.2)}$$

Properties of Error function

1. erf(-u) = -erf(u) - Symmetry.
2. erf(u) approaches unity as u tends towards infinity.

$$\frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-z^2) dz = 1 \qquad \text{-------- (A6.3)}$$

3. For a Random variable X, with mean m_x and variance $\sigma_x 2$, the probability of X is defined by

$$P(m_X - a < X \leq m_X + a) = erf\left(\frac{a}{\sqrt{2\sigma_X}}\right) \text{-------- (A6.4)}$$

Note: Relation: erfc(u) = 1 – erf(u)
Tables are used to find these values.

Approximate Relation: (only for large values of u)

$$erfc(u) < \frac{\exp(-u^2)}{\sqrt{\pi}} \qquad \text{-------- (A6.5)}$$

Q – Function:

An alternate form of error function. It basically defines the area under the Standardized Gaussian tail. For a standardized Gaussian random variable X of zero mean and unit variance, the Q-function is defined by

$$Q(v) = \frac{1}{\sqrt{2\pi}} \int_{v}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx$$ -------- (A6.6)

Relations between Q-function and erfc function:

(i) $$Q(v) = \frac{1}{2}\, erfc\left(\frac{v}{\sqrt{2}}\right)$$ ------- (A6.7a)

(ii) $$erfc(u) = 2Q(\sqrt{2}u)$$ ------ (A6.7b)

Probability of Error Calculation for Binary PCM Systems

Consider a binary communication system in which the two symbols binary1 and binary0 are represented by the signals $s_1(t)$ and $s_2(t)$ respectively. Let E_1 and E_2 represent the energies of the signals $s_1(t)$ and $s_2(t)$ respectively.

$$E_1 = \int_0^{Tb} s_1^2(t)\,dt \quad \text{and} \quad E_2 = \int_0^{Tb} s_2^2(t)\,dt \qquad \text{----- (A6.8)}$$

The Probability of error for Communication Systems can be defined as

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b\,(1-\rho)}{2N_0}}\right) \qquad \text{--------- (A6.9)}$$

Where E_b is the average energy per bit defined by

$$E_b = \frac{E_1 + E_2}{2} \qquad \text{--------- (A6.10)}$$

and ρ is the correlation coefficient

$$\rho = \frac{1}{E_b}\int_0^{Tb} s_1(t)s_2(t)\,dt \qquad \text{------ (A6.11)}$$

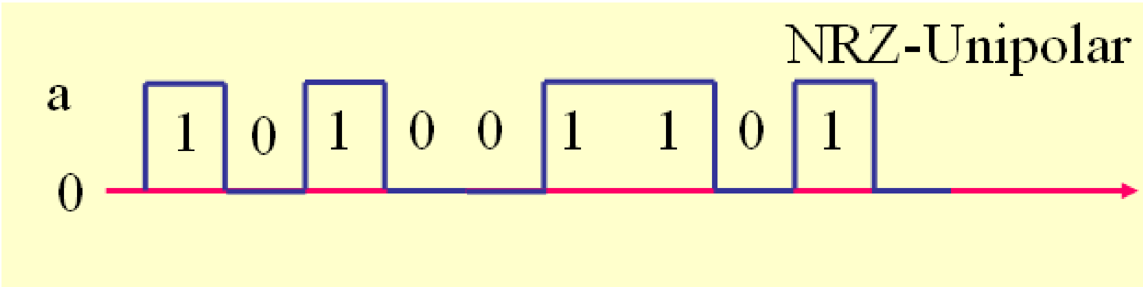
and (No/2) represent the noise power spectral density in W/Hz.

Case (1): Uni-polar signaling:

In this scheme the signals are represented as

$$S_1(t) = +a \qquad 0 \le t \le T_b \qquad for \quad Symbol\ 1$$

$$S_2(t) = 0 \qquad 0 \le t \le T_b \qquad for \quad Symbol\ 0$$



Signal energies are $E_1 = a^2 T_b$ and $E_2 = 0$

Average energy per bit, $E_b = a^2 T_b/2$.

Correlation coefficient = 0.

Probability of error,

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b(1-\rho)}{2N_0}}\right)$$

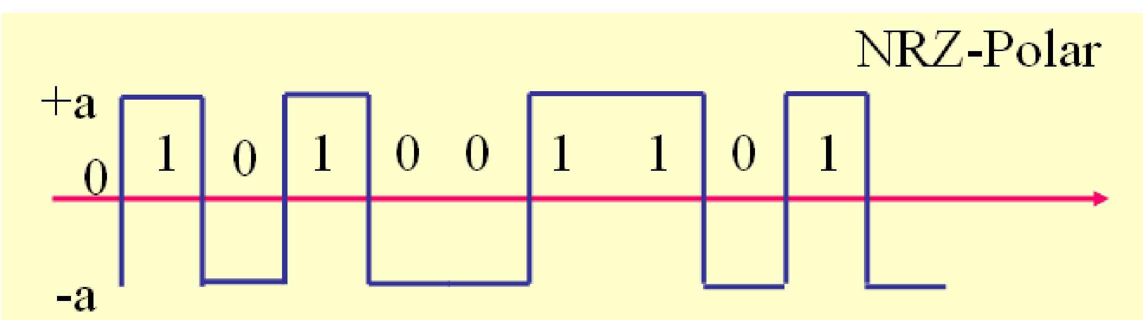
$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{a^2 T_b}{4N_0}}\right)$$

Case (2): Polar signaling:

In this scheme the signals are represented as

$$S_1(t) = +a \qquad 0 \le t \le T_b \qquad for \quad Symbol\ 1$$

$$S_2(t) = -a \qquad 0 \le t \le T_b \qquad for \quad Symbol\ 0$$



Signal energies are $E_1 = a^2 T_b$ and $E_2 = a^2 T_b$

Average energy per bit, $E_b = a^2 T_b$

Correlation coefficient = -1.

Probability of error,

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b(1-\rho)}{2N_0}}\right)$$

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{a^2 T_b}{N_0}}\right)$$

Case (3): Manchester signaling:

In this scheme the signals are represented as

$$\begin{aligned} S_1(t) &= +a/2 \qquad 0 \le t \le T_b/2 \qquad for \quad Symbol\ 1 \\ &\quad -a/2 \qquad T_b/2 < t < T_b \end{aligned}$$

$$\begin{aligned} S_2(t) &= -a/2 \qquad 0 \le t \le T_b/2 \qquad for \quad Symbol\ 0 \\ &\quad +a/2 \qquad T_b/2 < t < T_b \end{aligned}$$

Signal energies are $E_1 = a^2 T_b/4$ and $E_2 = a^2 T_b/4$

Average energy per bit, $E_b = a^2 T_b/4$

Correlation coefficient = -1.

Probability of error,

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b(1-\rho)}{2N_0}}\right)$$

Reduces to

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{a^2 T_b}{4N_0}}\right)$$

<u>Example:</u>

A binary PCM system using NRZ signaling operates just above the error threshold with an average probability of error equal to 10^{-6}. If the signaling rate is doubled, find the new value of the average probability of error.

Solution:

For probability of error equal to 10^{-6}.

$E_b/N_0 = 3.3$ (from table)

The probability of error is

$$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

If the signaling rate is doubled then E_b is reduced by a factor of 2 and correspondingly E_b/N_0 also reduces by 2. Hence the new probability of error will become .

$$P_e = 10^{-3}$$