Chapter-2

SAMPLING PROCESS

SAMPLING: A message signal may originate from a digital or analog source. If the message signal is analog in nature, then it has to be converted into digital form before it can transmitted by digital means. The process by which the continuous-time signal is converted into a discrete–time signal is called Sampling.

Sampling operation is performed in accordance with the sampling theorem.

Statement:- "If a band –limited signal g(t) contains no frequency components for $|f| > W$, then it is completely described by instantaneous values $g(kT_s)$ uniformly spaced in time with period $T_s \leq 1/2W$. If the sampling rate, fs is equal to the Nyquist rate or greater ($f_s \geq 2W$), the signal g(t) can be exactly reconstructed.

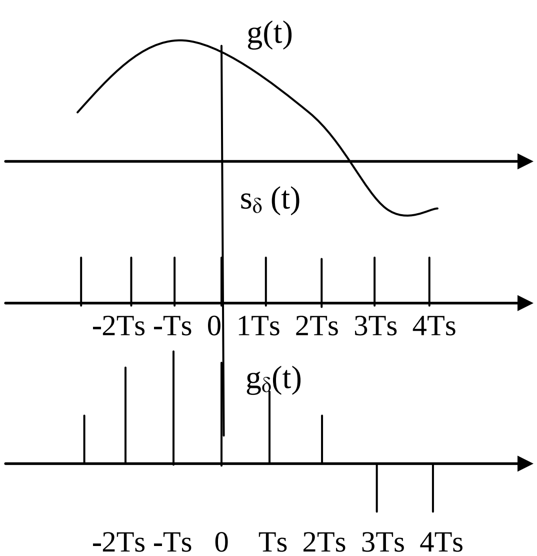


Fig 2.1: Sampling process

Proof:- Consider the signal g(t) is sampled by using a train of impulses $s_\delta(t)$.

Let $g_\delta(t)$ denote the ideally sampled signal, can be represented as

$$g_\delta(t) = g(t).s_\delta(t) \quad \text{-------------------} \quad 2.1$$

where $s_\delta(t)$ – impulse train defined by

$$s_\delta(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_s) \quad \text{--------------------} \quad 2.2$$

Therefore $\quad g_\delta(t) = g(t) . \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$

$$= \sum_{k=-\infty}^{+\infty} g(kT_s).\delta(t - kT_s) \quad \text{-----------} \quad 2.3$$

The Fourier transform of an impulse train is given by

SAMPLING THEOREM FOR LOW-PASS SIGNALS:-

$$S_\delta(f) = F[s_\delta(t)] = f_s \sum_{n=-\infty}^{+\infty} \delta(f - nf_s) \quad \text{------------} \quad 2.4$$

Applying F.T to equation 2.1 and using convolution in frequency domain property,

$$G_\delta(f) = G(f) * S_\delta(f)$$

Using equation 2.4, $\quad G_\delta(f) = G(f) * f_s \sum_{n=-\infty}^{+\infty} \delta(f - nf_s)$

$$G_\delta(f) = f_s \sum_{n=-\infty}^{+\infty} G(f - nf_s) \quad \text{-----------------} \quad 2.5$$

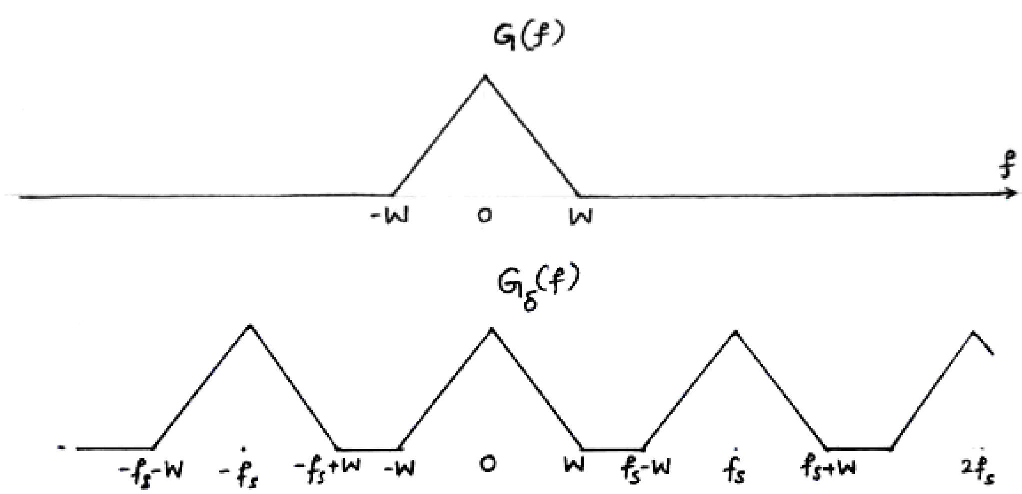


Fig. 2.2 Over Sampling (f_s > 2W)

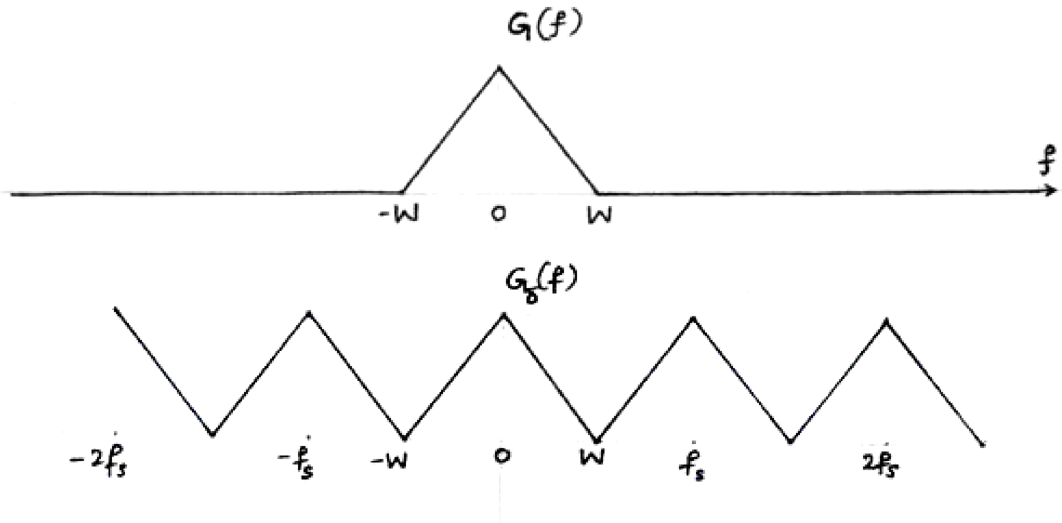


Fig. 2.3 Nyquist Rate Sampling (f_s = 2W)

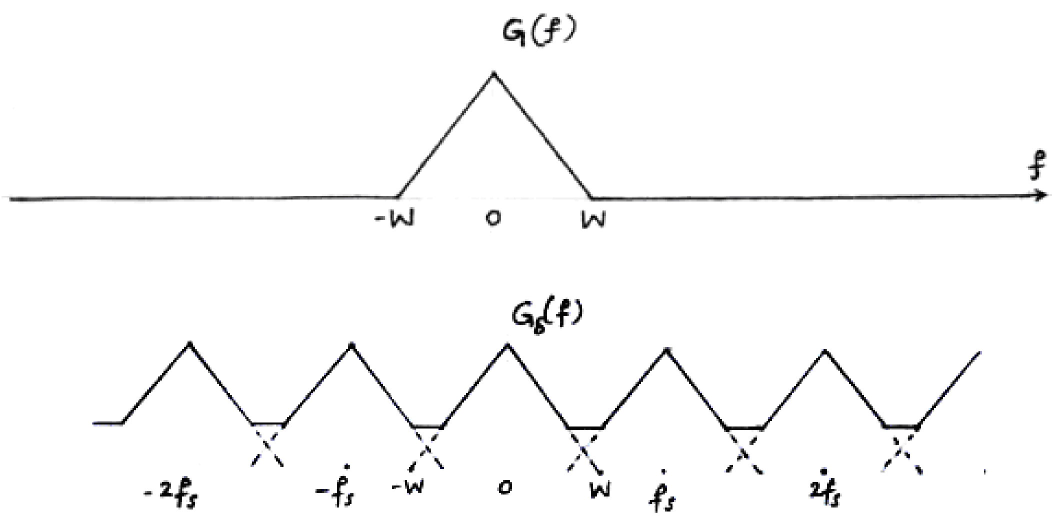
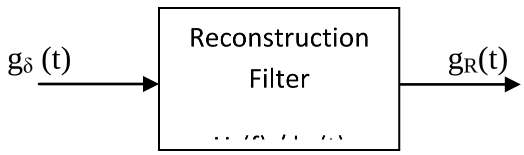


Fig. 2.4 Under Sampling (f_s < 2W)

Reconstruction of g(t) from g_δ (t):

By passing the ideally sampled signal $g_\delta(t)$ through an low pass filter (called Reconstruction filter) having the transfer function $H_R(f)$ with bandwidth, B satisfying the condition $W \le B \le (f_s - W)$, we can reconstruct the signal g(t). For an ideal reconstruction filter the bandwidth B is equal to W.



The output of LPF is, $g_R(t) = g_\delta(t) * h_R(t)$

where $h_R(t)$ is the impulse response of the filter.

In frequency domain, $G_R(f) = G_\delta(f) . H_R(f)$.

For the ideal LPF $H_R(f) = \begin{cases} K & -W \le f \le +W \\ 0 & \text{otherwise} \end{cases}$

then impulse response is $h_R(t) = 2WT_s . \text{Sinc}(2Wt)$

Correspondingly the reconstructed signal is

$$g_R(t) = [\, 2WT_s \text{ Sinc } (2Wt)] * [g_\delta (t)]$$

$$g_R(t) = 2WT_s \sum_{K=-\infty}^{+\infty} g(kTs).Sinc(2Wt) * \delta(t - kTs)$$

$$g_R(t) = 2WT_s \sum_{K=-\infty}^{+\infty} g(kTs).Sinc[2W(t - kTs)]$$

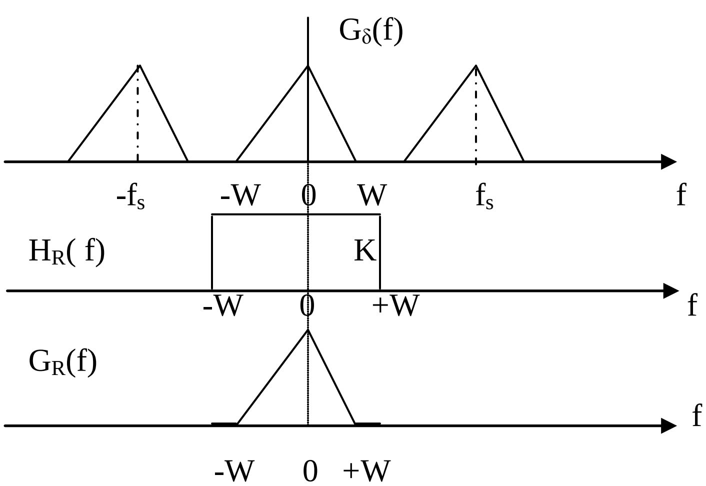


Fig: 2.5 Spectrum of sampled signal and reconstructed signal

Sampling of Band Pass Signals:

Consider a band-pass signal g(t) with the spectrum shown in figure 2.6:

Band width = B

Upper Limit = f_u

Lower Limit = f_l

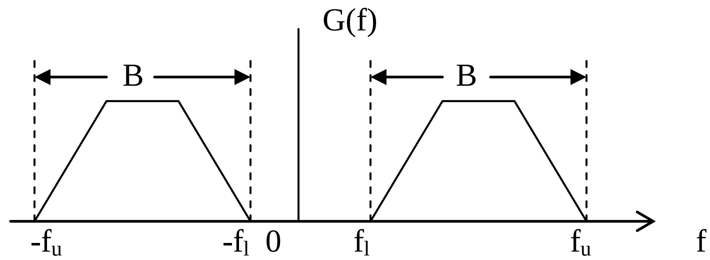


Fig 2.6: Spectrum of a Band-pass Signal

The signal g(t) can be represented by instantaneous values, g(kTs) if the sampling rate fs is $(2f_u/m)$ where m is an integer defined as

$$((f_u / B) - 1) < m \le (f_u / B)$$

If the sample values are represented by impulses, then g(t) can be exactly reproduced from it's samples by an ideal Band-Pass filter with the response, H(f) defined as

$$H(f) = \begin{cases} 1 & f_l < |f| < f_u \\ 0 & \text{elsewhere} \end{cases}$$

If the sampling rate, $fs \ge 2fu$, exact reconstruction is possible in which case the signal g(t) may be considered as a low pass signal itself.

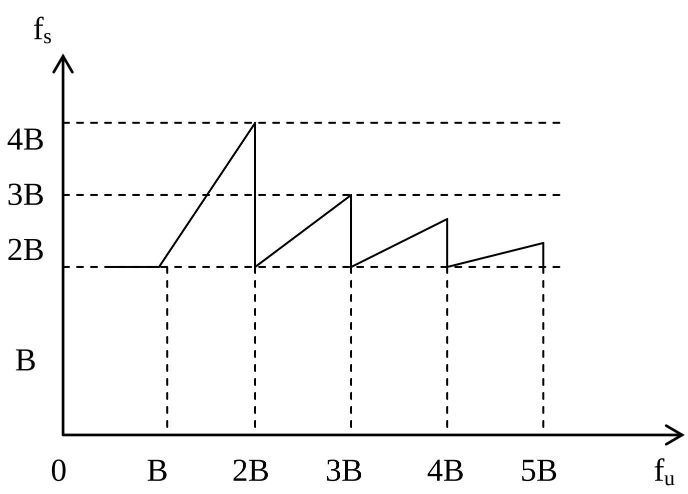


Fig 2.7: Relation between Sampling rate, Upper cutoff frequency and Bandwidth.

Example-1 :

Consider a signal g(t) having the Upper Cutoff frequency, $f_u = 100KHz$ and the Lower Cutoff frequency $f_l = 80KHz$.

The ratio of upper cutoff frequency to bandwidth of the signal g(t) is

$$f_u / B = 100K / 20K = 5.$$

Therefore we can choose m = 5.

Then the sampling rate is $f_s = 2f_u / m = 200K / 5 = 40KHz$

Example-2 :

Consider a signal g(t) having the Upper Cutoff frequency, $f_u = 120KHz$ and the Lower Cutoff frequency $f_l = 70KHz$.

The ratio of upper cutoff frequency to bandwidth of the signal g(t) is

$$f_u / B = 120K / 50K = 2.4$$

Therefore we can choose m = 2. ie.. m is an integer less than (f_u /B).

Then the sampling rate is $f_s = 2f_u / m = 240K / 2 = 120KHz$

Quadrature Sampling of Band – Pass Signals:

This scheme represents a natural extension of the sampling of low – pass signals.

In this scheme, the band pass signal is split into two components, one is in-phase component and other is quadrature component. These two components will be low–pass signals and are sampled separately. This form of sampling is called quadrature sampling.

Let g(t) be a band pass signal, of bandwidth '2W' centered around the frequency, fc, (fc>W). The in-phase component, $g_I(t)$ is obtained by multiplying g(t) with $\cos(2\pi f_c t)$ and then filtering out the high frequency components. Parallelly a quadrature phase component is obtained by multiplying g(t) with $\sin(2\pi f_c t)$ and then filtering out the high frequency components..

The band pass signal g(t) can be expressed as,

$$g(t) = g_I(t).\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$$

The in-phase, $g_I(t)$ and quadrature phase $g_Q(t)$ signals are low–pass signals, having band limited to $(-W < f < W)$. Accordingly each component may be sampled at the rate of 2W samples per second.

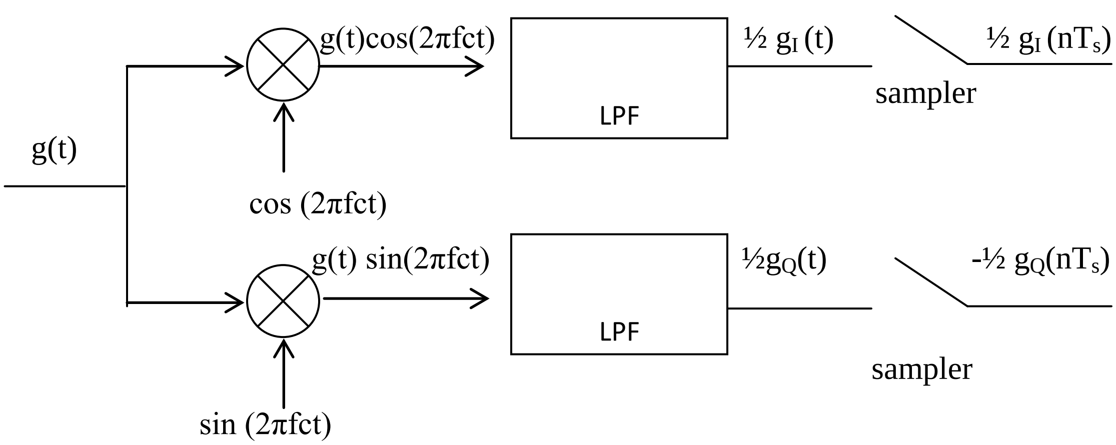
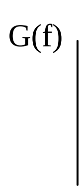
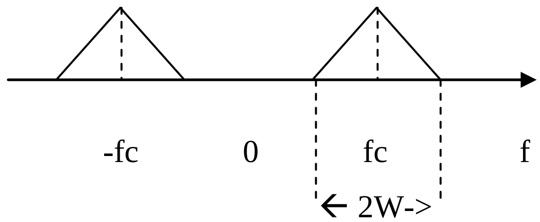
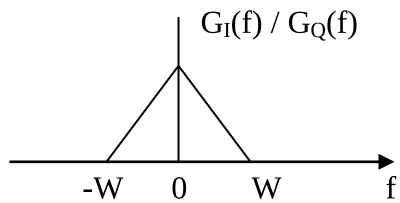


Fig 2.8: Generation of in-phase and quadrature phase samples





a) Spectrum of a Band pass signal.



b) Spectrum of $g_I(t)$ and $g_Q(t)$

Fig 2.9 a) Spectrum of Band-pass signal g(t)

b) Spectrum of in-phase and quadrature phase signals

RECONSTRUCTION:

From the sampled signals $g_I(nTs)$ and $g_Q(nTs)$, the signals $g_I(t)$ and $g_Q(t)$ are obtained. To reconstruct the original band pass signal, multiply the signals $g_I(t)$ by $\cos(2\pi fct)$ and $\sin(2\pi fct)$ respectively and then add the results.

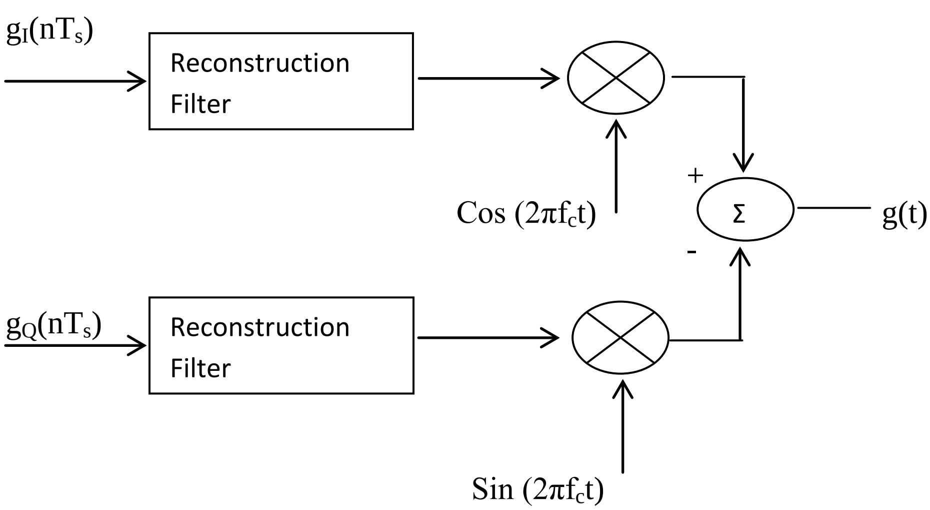
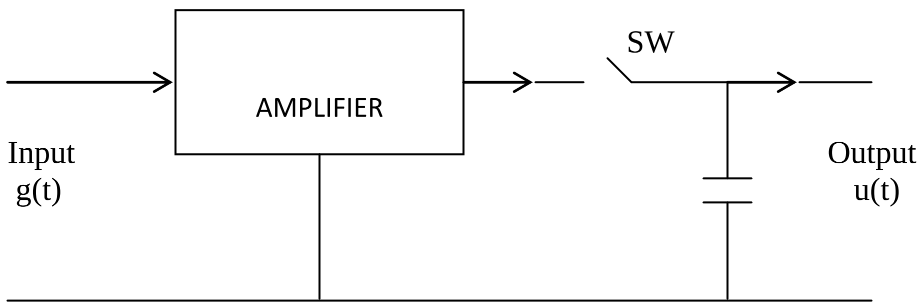


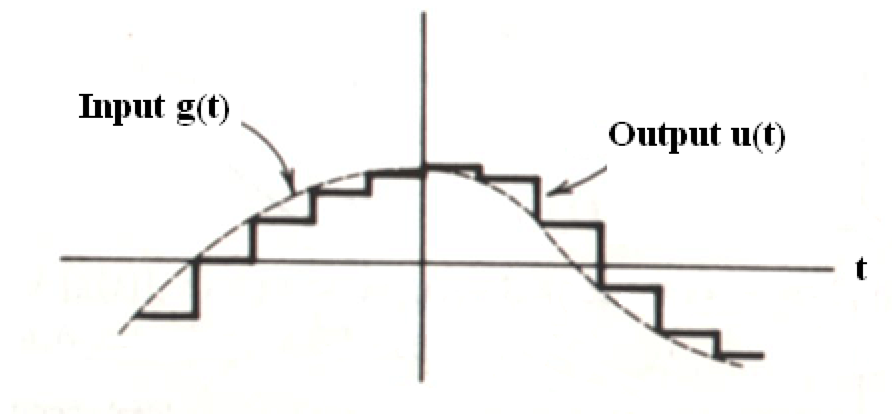
Fig 2.10: Reconstruction of Band-pass signal g(t)

Sample and Hold Circuit for Signal Recovery.

In both the natural sampling and flat-top sampling methods, the spectrum of the signals are scaled by the ratio τ/Ts, where τ is the pulse duration and Ts is the sampling period. Since this ratio is very small, the signal power at the output of the reconstruction filter is correspondingly small. To overcome this problem a sample-and-hold circuit is used .



a) Sample and Hold Circuit



b) Idealized output waveform of the circuit

Fig: 2.17 Sample Hold Circuit with Waveforms.

The Sample-and-Hold circuit consists of an amplifier of unity gain and low output impedance, a switch and a capacitor; it is assumed that the load impedance is large. The switch is timed to close only for the small duration of each sampling pulse, during which time the capacitor charges up to a voltage level equal to that of the input sample. When the switch is open , the capacitor retains the voltage level until the next closure of the switch. Thus the sample-and-hold circuit produces an output waveform that represents a staircase interpolation of the original analog signal.

The output of a Sample-and-Hold circuit is defined as

$$u(t) = \sum_{n=-\infty}^{+\infty} g(nTs)\, h(t - nTs)$$

where h(t) is the impulse response representing the action of the Sample-and-Hold circuit; that is

h(t) = 1 for 0 < t < Ts
0 for t < 0 and t > Ts

Correspondingly, the spectrum for the output of the Sample-and-Hold circuit is given by,

$$U(f) = f_s \sum_{n=-\infty}^{+\infty} H(f) G(f - nf_s))$$

where G(f) is the FT of g(t) and
H(f) = Ts Sinc(fTs) exp(-jπfTs)

To recover the original signal g(t) without distortion, the output of the Sample-and-Hold circuit is passed through a low-pass filter and an equalizer.

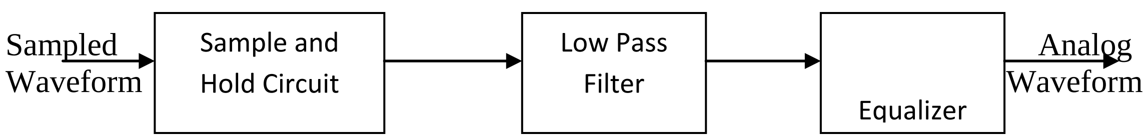


Fig. 2.18: Components of a scheme for signal reconstruction

Signal Distortion in Sampling.

In deriving the sampling theorem for a signal g(t) it is assumed that the signal g(t) is strictly band-limited with no frequency components above 'W' Hz. However, a signal cannot be finite in both time and frequency. Therefore the signal g(t) must have infinite duration for its spectrum to be strictly band-limited.

In practice, we have to work with a finite segment of the signal in which case the spectrum cannot be strictly band-limited. Consequently when a signal of finite duration is sampled an error in the reconstruction occurs as a result of the sampling process.

Consider a signal g(t) whose spectrum G(f) decreases with the increasing frequency without limit as shown in the figure 2.19. The spectrum, $G_\delta(f)$ of the ideally sampled signal , $g_\delta(t)$ is the sum of G(f) and infinite number of frequency shifted replicas of G(f). The replicas of G(f) are shifted in frequency by multiples of sampling frequency, fs. Two replicas of G(f) are shown in the figure 2.19.

The use of a low-pass reconstruction filter with it's pass band extending from (-fs/2 to +fs/2) no longer yields an undistorted version of the original signal g(t). The portions of the frequency shifted replicas are folded over inside the desired spectrum. Specifically, high frequencies in G(f) are reflected into low frequencies in $G_\delta(f)$. The phenomenon of overlapping in the spectrum is called as Aliasing or Foldover Effect. Due to this phenomenon the information is invariably lost.

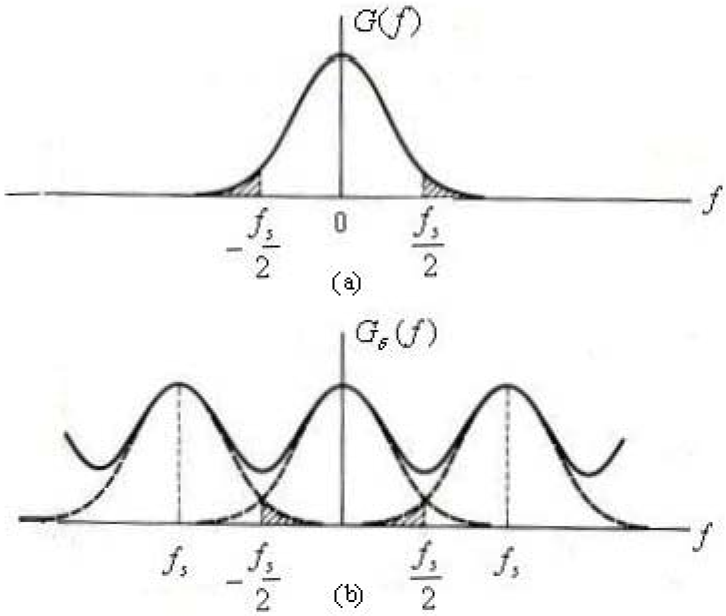


Fig. 2.19 : a) Spectrum of finite energy signal g(t)
b) Spectrum of the ideally sampled signal.

Bound On Aliasing Error:

Let g(t) be the message signal, g(n/fs) denote the sequence obtained by sampling the signal g(t) and $g_i(t)$ denote the signal reconstructed from this sequence by interpolation; that is

$$g_i(t) = \sum_n g\left(\frac{n}{f_s}\right) Sinc(f_s t - n)$$

Aliasing Error is given by, ε = | g(t) - gi(t) |

Signal g(t) is given by

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df$$

Or equivalently

$$g(t) = \sum_{m=-\infty}^{+\infty} \int_{(m-1/2)fs}^{(m+1/2)fs} G(f)\exp(j2\pi ft)df$$

Using Poisson's formula and Fourier Series expansions we can obtain the aliasing error as

$$\varepsilon = \left| \sum_{m=-\infty}^{+\infty} [1-\exp(-j2\pi m f_s t)] \int_{(m-1/2)fs}^{(m+1/2)fs} G(f)\exp(j2\pi ft)df \right|$$

Correspondingly the following observations can be done :

1. The term corresponding to m=0 vanishes.
2. The absolute value of the sum of a set of terms is less than or equal to the sum of the absolute values of the individual terms.
3. The absolute value of the term 1- exp(-j2πmfst) is less than or equal to 2.
4. The absolute value of the integral in the above equation is bounded as

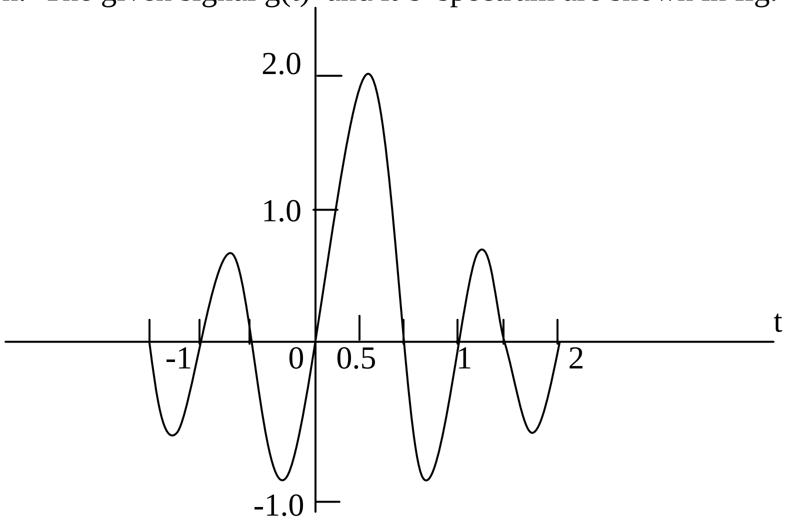
$$\left| \int_{(m-1/2)fs}^{(m+1/2)fs} G(f)\exp(j2\pi ft)df \right| < \int_{(m-1/2)fs}^{(m+1/2)fs} |G(f)|\,df$$

Hence the aliasing error is bounded as

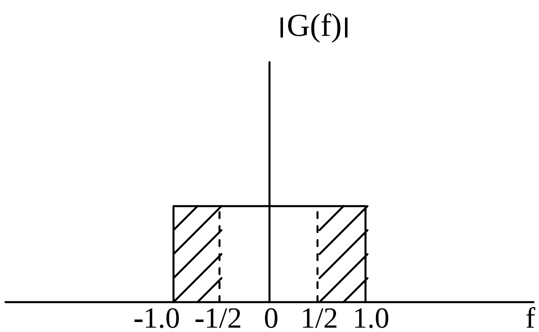
$$\varepsilon \leq 2\int_{|f|>fs/2} |G(f)|\,df$$

Example: Consider a time shifted sinc pulse, g(t) = 2 sinc(2t – 1). If g(t) is sampled at rate of 1sample per second that is at t = 0, ± 1, ±2, ±3 and so on , evaluate the aliasing error.

Solution: The given signal g(t) and it's spectrum are shown in fig. 2.20.



a) Sinc Pulse



(b) Amplitude Spectrum, |G(f)|

Fig. 2.20

The sampled signal g(nTs) = 0 for n = 0, ± 1, ±2, ±3 . . . . .and reconstructed signal

$g_i(t) = 0$ for all t.

From the figure, the sinc pulse attains it's maximum value of 2 at time t equal to ½. The aliasing error cannot exceed max|g(t)| = 2.

From the spectrum, the aliasing error is equal to unity.

Natural Sampling:

In this method of sampling, an electronic switch is used to periodically shift between the two contacts at a rate of fs = (1/Ts) Hz, staying on the input contact for C seconds and on the grounded contact for the remainder of each sampling period.

The output $x_s(t)$ of the sampler consists of segments of x(t) and hence $x_s(t)$ can be considered as the product of x(t) and sampling function s(t).

$$x_s(t) = x(t) \cdot s(t)$$

The sampling function s(t) is periodic with period Ts, can be defined as,

$$S(t) = \begin{cases} 1 & -\tau/2 < t < \tau/2 \\ 0 & \tau/2 < |t| < Ts/2 \end{cases} \quad \text{------- (1)}$$

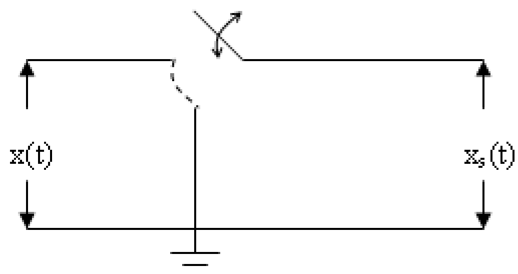


Fig: 2.11 Natural Sampling – Simple Circuit.

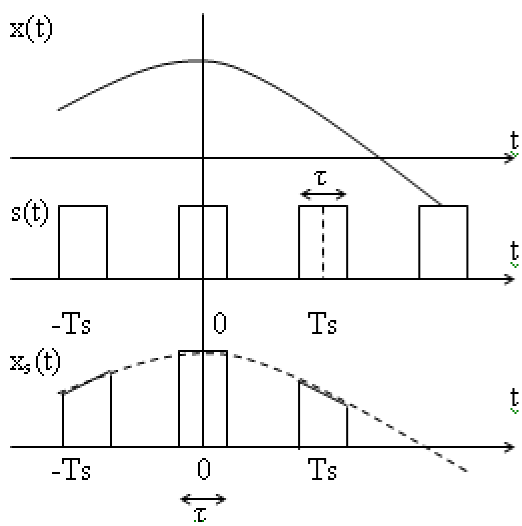


Fig: 2.12 Natural Sampling – Waveforms.

Using Fourier series, we can rewrite the signal S(t) as

$$S(t) = Co + \sum_{n=1}^{\infty} 2Cn\cos(nw_s t)$$

where the Fourier coefficients, $Co = \tau / Ts$ & $Cn = fs\tau\, \text{Sinc}(n\, fs\tau)$

Therefore: $x_s(t) = x(t)\left[Co + \sum_{n=1}^{\infty} 2Cn\cos(nw_s t) \right]$

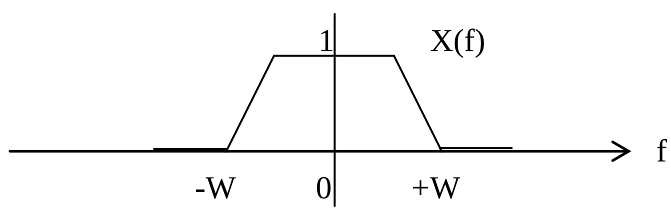
$x_s(t) = Co.x(t) + 2C_1.x(t)\cos(w_s t) + 2C_2.x(t)\cos(2w_s t) + \ldots\ldots..$

Applying Fourier transform for the above equation

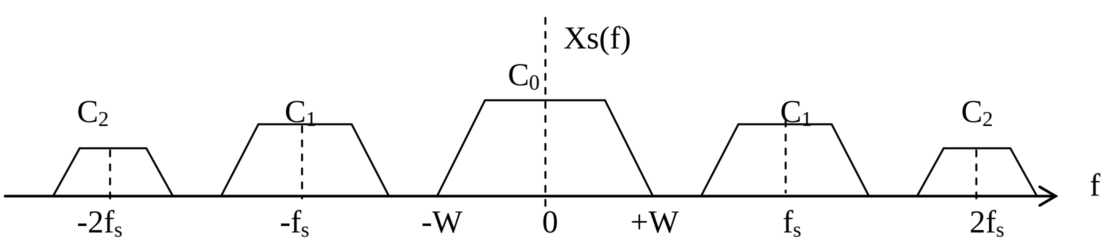
$$\begin{cases} \text{Using } x(t) \overset{FT}{\longleftrightarrow} X(f) \\ x(t)\cos(2\pi f_0 t) \longleftrightarrow \tfrac{1}{2}[X(f-f_0) + X(f+f_0)] \end{cases}$$

$$Xs(f) = Co.X(f) + C_1[X(f-f_0) + X(f+f_0)] + C_2[X(f-f_0) + X(f+f_0)] + \ldots\ldots$$

$$Xs(f) = Co.X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} Cn.X(f - nfs)$$



Message Signal Spectrum



Sampled Signal Spectrum ($f_s > 2W$)

Fig:2.13 Natural Sampling Spectrum

The signal $x_s(t)$ has the spectrum which consists of message spectrum and repetition of message spectrum periodically in the frequency domain with a period of f_s. But the message term is scaled by 'Co". Since the spectrum is not distorted it is possible to reconstruct x(t) from the sampled waveform $x_s(t)$.

Flat Top Sampling:

In this method, the sampled waveform produced by practical sampling devices, the pulse p(t) is a flat – topped pulse of duration, τ .

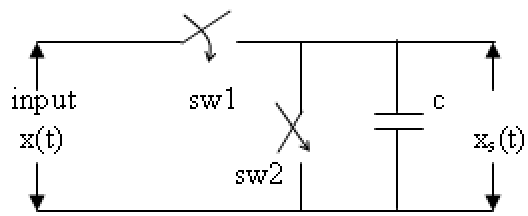


Fig. 2.14: Flat Top Sampling Circuit

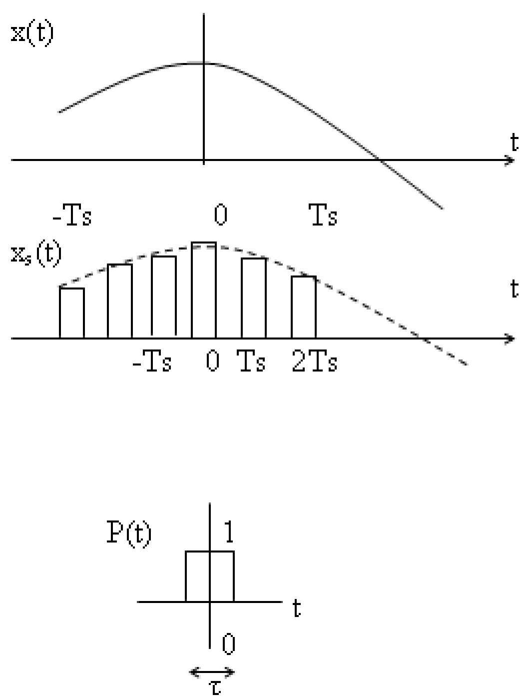


Fig. 2.15: Waveforms

Mathematically we can consider the flat – top sampled signal as equivalent to the convolved sequence of the pulse signal p(t) and the ideally sampled signal, $x_\delta(t)$.

$$x_s(t) = p(t) * x_\delta(t)$$

$$x_s(t) = p(t) * [\sum_{k=-\infty}^{+\infty} x(kTs).\delta(t - kTs)]$$

Applying F.T,

$$X_s(f) = P(f).X_\delta(f)$$

$$= P(f).\ \ fs \sum_{n=-\infty}^{+\infty} X(f - nfs)$$

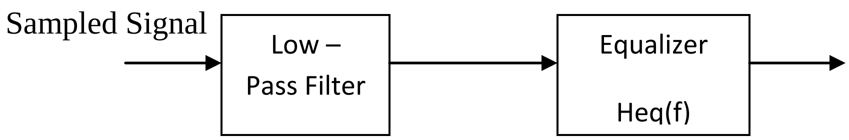
where P(f) = FT[p(t)] and $X_\delta(f) = FT[x_\delta(t)]$

Aperature Effect:

The sampled signal in the flat top sampling has the attenuated high frequency components. This effect is called the Aperture Effect.

The aperture effect can be compensated by:

1. Selecting the pulse width τ as very small.
2. by using an equalizer circuit.



Equalizer decreases the effect of the in-band loss of the interpolation filter (lpf).
As the frequency increases, the gain of the equalizer increases. Ideally the amplitude response of the equalizer is

$$| H_{eq}(f) | = 1 / | P(f) | = \frac{1}{\tau.SinC(f\tau)} = \frac{\pi f}{Sin(\pi f \tau)}$$